Lecture 4

Math 17B
Laplace Transforms 4
HW # 21

Laplace Transforms Applied to Systems of Differential Equations

Def: Let $x_1 = f(t)$ and $x_2 = g(t)$ be differentiable functions of t. Then

$$\begin{cases} \dfrac{dx_1}{dt} = a \cdot x_1 + b \cdot x_2 \\ \dfrac{dx_2}{dt} = c \cdot x_1 + d\, x_2 \end{cases}$$

is called a 2x2 linear system of D.E.'s, where a, b, c, and d are constants.

Note: Systems of D.E.'s can be used to describe mixture problems, spring problems, and predator/prey problems.

Ex: Use Laplace transforms to the following systems of D.E.'s.

1.) $$\begin{cases} \dfrac{dx_1}{dt} = x_2 , & x_1(0) = 0 \\ \dfrac{dx_2}{dt} = -x_1 , & x_2(0) = 1 \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\left\{\frac{dx_1}{dt}\right\} = \mathcal{L}\{x_2\} \\ \mathcal{L}\left\{\frac{dx_2}{dt}\right\} = -\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \begin{cases} s\mathcal{L}\{x_1\} - \cancel{x_1(0)}^{\,0} = \mathcal{L}\{x_2\} \\ s\mathcal{L}\{x_2\} - \cancel{x_2(0)}^{\,1} = -\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\{x_2\} = s\mathcal{L}\{x_1\} \longrightarrow (\text{SUB}) \\ s\mathcal{L}\{x_2\} - 1 = -\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \quad s\left(s\mathcal{L}\{x_1\}\right) - 1 = -\mathcal{L}\{x_1\}$$

$$\rightarrow \quad (s^2+1)\mathcal{L}\{x_1\} = 1$$

$$\rightarrow \quad \mathcal{L}\{x_1\} = \frac{1}{s^2+1} \longrightarrow \boxed{x_1 = \sin t} \;;$$

and $\mathcal{L}\{x_2\} = s\mathcal{L}\{x_1\} = s \cdot \frac{1}{s^2+1} = \frac{s}{s^2+1}$

$$\rightarrow \boxed{x_2 = \cos t}\,.$$

2.) $\begin{cases} \frac{dx_1}{dt} = x_1 + x_2 \quad , \quad x_1(0) = 3 \\ \frac{dx_2}{dt} = 2x_1 \quad , \quad x_2(0) = 0 \end{cases}$

$$\rightarrow \begin{cases} \mathcal{L}\left\{\frac{dx_1}{dt}\right\} = \mathcal{L}\{x_1 + x_2\} \\ \mathcal{L}\left\{\frac{dx_2}{dt}\right\} = \mathcal{L}\{2x_1\} \end{cases}$$

$$\rightarrow \begin{cases} s\mathcal{L}\{x_1\} - \cancel{x_1(0)}^{\,3} = \mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \\ s\mathcal{L}\{x_2\} - \cancel{x_2(0)}^{\,0} = 2\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\{x_2\} = (s-1)\mathcal{L}\{x_1\} - 3 \rightarrow (SUB) \\ s\mathcal{L}\{x_2\} = 2\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \quad s\big((s-1)\mathcal{L}\{x_1\} - 3\big) = 2\mathcal{L}\{x_1\}$$

$$\rightarrow \quad s(s-1)\mathcal{L}\{x_1\} - 3s = 2\mathcal{L}\{x_1\}$$

$$\rightarrow \quad (s^2 - s - 2)\mathcal{L}\{x_1\} = 3s$$

$$\rightarrow \quad \mathcal{L}\{x_1\} = \frac{3s}{s^2 - s - 2} = \frac{3s}{(s-2)(s+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+1} \rightarrow A(s+1) + B(s-2) = 3s:$$

Let $s = 2$: $3A = 6 \rightarrow A = 2$,

Let $s = -1$: $-3B = -3 \rightarrow B = 1$; then

$$\mathcal{L}\{x_1\} = \frac{2}{s-2} + \frac{1}{s+1} \rightarrow$$

$$x_1 = 2e^{2t} + e^{-t}$$; and

$$s\mathcal{L}\{x_2\} = 2\mathcal{L}\{x_1\} \rightarrow$$

$$s\mathcal{L}\{x_2\} = 2\left(\frac{2}{s-2} + \frac{1}{s+1}\right) \rightarrow$$

$$\mathcal{L}\{x_2\} = \frac{4}{s(s-2)} + \frac{2}{s(s+1)}$$

$$\left(\frac{4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \rightarrow A(s-2) + Bs = 4:\right.$$

$\underline{\text{Let } s=0}$: $-2A = 4 \rightarrow A = -2$,

$\underline{\text{Let } s=2}$: $2B = 4 \rightarrow B = 2$; and

$$\frac{2}{s(s+1)} = \frac{C}{s} + \frac{D}{s+1} \rightarrow C(s+1) + Ds = 2:$$

$\underline{\text{Let } s=0}$: $C = 2$,

$\underline{\text{Let } s=-1}$: $-D = 2 \rightarrow D = -2 \Big) \rightarrow$

$$\mathcal{L}\{x_2\} = \left(\frac{-2}{s} + \frac{2}{s-2}\right) + \left(\frac{2}{s} + \frac{-2}{s+1}\right) \rightarrow$$

$$\mathcal{L}\{x_2\} = \frac{2}{s-2} - \frac{2}{s+1} \rightarrow$$

$$x_2 = 2e^{2t} - 2e^{-t}$$

3.) $$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2, & x_1(0) = 0 \\ \dfrac{dx_2}{dt} = -x_1 + x_2, & x_2(0) = 1 \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\left\{\dfrac{dx_1}{dt}\right\} = \mathcal{L}\{x_1 + x_2\} \\ \mathcal{L}\left\{\dfrac{dx_2}{dt}\right\} = \mathcal{L}\{-x_1 + x_2\} \end{cases}$$

$$\rightarrow \begin{cases} s\mathcal{L}\{x_1\} - \cancelto{0}{x_1(0)} = \mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \\ s\mathcal{L}\{x_2\} - \cancelto{1}{x_2(0)} = -\mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \end{cases}$$

$$\rightarrow \mathcal{L}\{x_2\} = (s-1)\mathcal{L}\{x_1\} \rightarrow \text{(SUB)}$$

$\longrightarrow s\left((s-1)\mathcal{L}\{x_1\}\right) - 1 = -\mathcal{L}\{x_1\} + (s-1)\mathcal{L}\{x_1\}$

$\longrightarrow (s^2 - s)\mathcal{L}\{x_1\} + \mathcal{L}\{x_1\} - (s-1)\mathcal{L}\{x_1\} = 1$

$\longrightarrow (s^2 - s + 1 - (s-1))\mathcal{L}\{x_1\} = 1$

$\longrightarrow (s^2 - 2s + 2)\,\mathcal{L}\{x_1\} = 1$

$\longrightarrow \mathcal{L}\{x_1\} = \dfrac{1}{s^2 - 2s + 2} = \dfrac{1}{(s-1)^2 + 1^2}$

$\longrightarrow$ $x_1 = e^t \sin t$; and

$\mathcal{L}\{x_2\} = (s-1)\,\mathcal{L}\{x_1\} = (s-1)\cdot\dfrac{1}{(s-1)^2+1} \longrightarrow$

$\mathcal{L}\{x_2\} = \dfrac{s-1}{(s-1)^2 + 1^2} \longrightarrow$

$x_2 = e^t \cos t$