Math 17B
Kouba
Using Eigenvalues and Eigenvectors to Solve Matrix Algebra Problems

DERIVE A FORMULA FOR POWERS OF A MATRIX APPLIED TO ITS EIGENVECTOR :

Let A be a matrix and assume that λ is an eigenvalue with eigenvector V . Then

$$AV = \lambda V \quad \longrightarrow$$

$$A^2V = A(AV) = A(\lambda V) = \lambda(AV) = \lambda(\lambda V) = \lambda^2 V \quad \longrightarrow$$

$$A^3V = A(A^2V) = A(\lambda^2 V) = \lambda^2(AV) = \lambda^2(\lambda V) = \lambda^3 V \quad \longrightarrow$$

$$A^4V = A(A^3V) = A(\lambda^3 V) = \lambda^3(AV) = \lambda^3(\lambda V) = \lambda^4 V \quad \longrightarrow$$

$$\text{(P)} \qquad A^kV = \lambda^k V \quad \text{for} \quad k = 1, 2, 3, 4, \cdots$$

USE THE ABOVE FORMULA TO SOLVE THE FOLLOWING PROBLEM :

EXAMPLE : Consider matrix $A = \begin{pmatrix} -1/2 & 5/4 \\ 1 & 3/2 \end{pmatrix}$. It can be shown that it's eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$ with eigenvectcors $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $V_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, resp.

Use the eigenvalues and eigenvectors for matrix A to determine $A^{30}\begin{pmatrix} 13 \\ 2 \end{pmatrix}$.

Begin by writing vector $\begin{pmatrix} 13 \\ 2 \end{pmatrix}$ as a linear combination of the eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$:

$$\begin{pmatrix} 13 \\ 2 \end{pmatrix} = (c_1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_2)\begin{pmatrix} -5 \\ 2 \end{pmatrix} \quad \longrightarrow$$

$$\begin{aligned} 13 &= c_1 - 5c_2 \\ 2 &= 2c_1 + 2c_2 \quad \longrightarrow \end{aligned}$$

$$\begin{aligned} -26 &= -2c_1 + 10c_2 \\ 2 &= 2c_1 + 2c_2 \quad \longrightarrow \quad -24 = 12c_2 \quad \longrightarrow \quad c_2 = -2 \ \text{ and } \ c_1 = 3 \ . \end{aligned}$$

Thus, we have $\begin{pmatrix} 13 \\ 2 \end{pmatrix} = (3)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-2)\begin{pmatrix} -5 \\ 2 \end{pmatrix} \quad \longrightarrow$

$$A^{30}\begin{pmatrix}13\\2\end{pmatrix} = A^{30}\left((3)\begin{pmatrix}1\\2\end{pmatrix} + (-2)\begin{pmatrix}-5\\2\end{pmatrix}\right)$$

$$= A^{30}\left((3)\begin{pmatrix}1\\2\end{pmatrix}\right) + A^{30}\left((-2)\begin{pmatrix}-5\\2\end{pmatrix}\right)$$

$$= (3)\left(A^{30}\begin{pmatrix}1\\2\end{pmatrix}\right) + (-2)\left(A^{30}\begin{pmatrix}-5\\2\end{pmatrix}\right)$$

(Now use equation (P).)

$$= (3)\left((2)^{30}\begin{pmatrix}1\\2\end{pmatrix}\right) + (-2)\left((-1)^{30}\begin{pmatrix}-5\\2\end{pmatrix}\right)$$

$$= \begin{pmatrix}(3)(2)^{30}\\(3)(2)^{31}\end{pmatrix} + \begin{pmatrix}10\\-4\end{pmatrix}$$

$$= \begin{pmatrix}3,221,225,482\\6,442,450,940\end{pmatrix}.$$