Math 17B
Kouba
Estimating the Value of a Definite Integral

Suppose that the integral $\int_a^b f(x)\,dx$ is too difficult (or impossible) to compute, or that you are simply required to estimate its exact value. The following two methods offer two different ways to determine good estimates.

1.) MIDPOINT RULE

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval and let the sampling points $c_1, c_2, c_3, ..., c_n$ be the MIDPOINTS of these subintervals.

c.) The Midpoint Estimate for $\int_a^b f(x)\,dx$ is

$$M_n = h\ [f(c_1) + f(c_2) + f(c_3) + ... + f(c_n)] \ .$$

d.) The Absolute Error $|E_n| = \left| \int_a^b f(x)\,dx - M_n \right|$ satisfies

$$|E_n| \le (b-a)\frac{h^2}{24}\left\{ \max_{a \le x \le b} |f''(x)| \right\} .$$

2.) TRAPEZOIDAL RULE

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Trapezoidal Estimate for $\int_a^b f(x)\,dx$ is

$$T_n = \frac{h}{2}\ [f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)] \ .$$

d.) The Absolute Error $|E_n| = \left| \int_a^b f(x)\,dx - T_n \right|$ satisfies

$$|E_n| \le (b-a)\frac{h^2}{12}\left\{ \max_{a \le x \le b} |f''(x)| \right\} .$$

Ex: Determine the value of n so that the Trapezoidal Estimate T_n estimates the exact value of $\int_{-1}^{2} \frac{1}{\sqrt{x+5}}\,dx$ with absolute error at most 0.0001:

Let $f(x) = \frac{1}{\sqrt{x+5}} = (x+5)^{-1/2}$ on $[-1,2]$. Then

$$h = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n} \quad \text{and} \quad f'(x) = -\frac{1}{2}(x+5)^{-3/2}$$

$$\xrightarrow{D} f''(x) = \frac{3}{4}(x+5)^{-5/2} = \frac{3}{4(x+5)^{5/2}} \;;$$

$$\max_{-1\le x\le 2} |f''(x)| = \max_{-1\le x\le 2} \left| \frac{3}{4(x+5)^{5/2}} \right|$$

$$= \max_{-1\le x\le 2} \frac{3}{4|x+5|^{5/2}} = \frac{3}{4|(-1)+5|^{5/2}} = \frac{3}{128} \;; \text{ so}$$

$$|E_n| \le (b-a)\cdot\frac{h^2}{12}\cdot\left\{\max_{-1\le x\le 2}|f''(x)|\right\}$$

$$= (2-(-1))\cdot\frac{1}{12}\left(\frac{3}{n}\right)^2\cdot\left\{\frac{3}{128}\right\}$$

$$= (3)\cdot\frac{1}{12}\cdot\frac{9}{n^2}\cdot\left\{\frac{3}{128}\right\}$$

$$= \frac{1}{4}\cdot\frac{9}{n^2}\cdot\left\{\frac{3}{128}\right\} = \frac{27}{512}\cdot\frac{1}{n^2} \le 0.0001 \rightarrow$$

$$n^2 \ge \frac{27}{512(0.0001)} \rightarrow n \ge \sqrt{\frac{27}{512(0.0001)}} \approx 22.9$$

so choose $\boxed{n = 23}$ (or bigger)