Math 17B
Kouba
Rotation Matrices in Two-Dimensional Space

DEFINITION : A matrix of the form $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is called a *rotation matrix.*

FACT : If $\theta > 0$, then R rotates vectors θ radians counter-clockwise. If $\theta < 0$, then R rotates vectors $|\theta|$ radians clockwise. Note that if $\theta = 0$, then $R = I_2$, the two-by-two identity matrix.

PROOF : (For $\theta > 0$) Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a random vector in two-dimensional space with polar form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$, where r is the length and α is the direction of the vector . We want to investigate the vector $R\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$\begin{aligned}
R\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= R\begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \\
&= \begin{pmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ r(\cos\alpha\sin\theta + \sin\alpha\cos\theta) \end{pmatrix} \\
&= \begin{pmatrix} r\cdot\cos(\alpha+\theta) \\ r\cdot\sin(\alpha+\theta) \end{pmatrix} .
\end{aligned}$$

This new vector has length r and direction $\alpha + \theta$. Thus, matrix R has rotated vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ θ radians counter-clockwise.