Math 17B
Kouba
Taylor Polynomials

QUESTION : Consider the function $y = f(x)$ and assume that mathematical circumstances would require that this function be replaced with an nth-degree polynomial whose values closely approximate the values of $y = f(x)$. How would we determine the coefficients of this unknown polynomial ?

ANSWER : Assume that $y = f(x)$ is a given function and constant "a" is known. Determine a list of real numbers $a_0, a_1, a_2, a_3, \cdots, a_n$ so that the polynomial is

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n \ ,$$

which satisfies the condition

$$f(x) \approx P_n(x) \ ,$$

i.e.,

$$\text{(T)} \qquad f(x) \approx a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n \ .$$

If we assume that " $\approx$ " is " $=$" and substitute $x = a$ in equation (T), we get

$$f(a) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots + a_n(0)^n = a_0 \ ,$$

i.e.,

$$a_0 = f(a) \ .$$

Now differentiate equation (T) term by term getting

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots + na_n(x-a)^{n-1} \ .$$

If we substitute $x = a$ in this equation, we get

$$f'(a) = a_1 + 2a_2(0) + 3a_3(0)^2 + 4a_4(0)^3 + \cdots + na_n(0)^{n-1} = a_1 \ ,$$

i.e.,

$$a_1 = f'(a) \ .$$

Now differentiate again term by term getting

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 + 5 \cdot 4a_5(x-a)^3 + \cdots + n \cdot (n-1)a_n x^{n-2}.$$

If we substitute $x = a$ in this equation, we get

$$f''(a) = 2a_2 + 3 \cdot 2a_3(0) + 4 \cdot 3a_4(0)^2 + 5 \cdot 4a_4(0)^3 + \cdots + (n-1) \cdot na_n(0)^{n-2} = 2a_2 \ ,$$

i.e.,

$$a_2 = \frac{f''(a)}{2!} \ .$$

Now differentiate again term by term getting

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-a) + 5 \cdot 4 \cdot 3a_5(x-a)^2 + \cdots + n(n-1)(n-2)a_n x^{n-3} \ .$$

If we substitute $x = a$ in this equation, we get

$$f'''(a) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(0) + 5\cdot 4\cdot 3a_5(0)^2 + \cdots n(n-1)(n-2)a_n(0)^{n-3} = 3\cdot 2a_3 \ ,$$

i.e.,

$$a_3 = \frac{f'''(a)}{3!} \ .$$

Continuing this term by term differentiation and substitution process results in the fact that

(S) $$a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, 2, 3, \cdots, n \ .$$

<u>DEFINITION</u> : The *Taylor Polynomial of degree* n centered at $x = a$ for function $y = f(x)$ is given by

(P) $$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$$

and

(S) $$a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, 2, 3, \cdots, n \ .$$

<u>QUESTION</u> : Consider an ordinary function $y = f(x)$ and $P_n(x)$, its Taylor Polynomial of degree n centered at $x = a$. We assume that $f(x) \approx P_n(x)$. Let the error be given by $R_{n+1}(x;a) = f(x) - P_n(x)$. It can be shown that the absolute error (Taylor Error) for their difference is

$$\left| R_{n+1}(x;a) \right| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| ,$$

where c is some number between x and a. (Note that the number c appears as a result of using the Intermediate Value Theorem to derive this error formula.)