Math 17B (Spring 2017)
Kouba
Exam 2

Your Name : KEY

Your EXAM ID Number ___________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You will be graded on proper use of integral and derivative notation.

7. You will be graded on proper use of limit notation.

8. You have until 10:52 a.m. sharp to finish the exam.

1.) (9 pts. each) Solve the following differential equations.

a.) $\frac{dy}{dx} = x^{1/2} + e^{2x} - \frac{1}{x^3} + 5 \quad = x^{1/2} + e^{2x} - x^{-3} + 5$

$\rightarrow Y = \int (x^{1/2} + e^{2x} - x^{-3} + 5)\, dx$

$\rightarrow Y = \frac{2}{3}x^{3/2} + \frac{1}{2}e^{2x} - \frac{-1}{2}x^{-2} + 5x + C$

b.) $\frac{dy}{dx} = \frac{xy}{y^2+1} \rightarrow \int \frac{y^2+1}{Y}\, dy = \int x\, dx \rightarrow$

$\int \left(Y + \frac{1}{Y}\right) dy = \frac{1}{2}x^2 + C \rightarrow$

$\frac{1}{2}Y^2 + \ln|Y| = \frac{1}{2}x^2 + C$

c.) $x\frac{dy}{dx} + y = x^2 + \cos x \longrightarrow Y' + \frac{1}{x}Y = \frac{x^2 + \cos x}{x} \rightarrow$

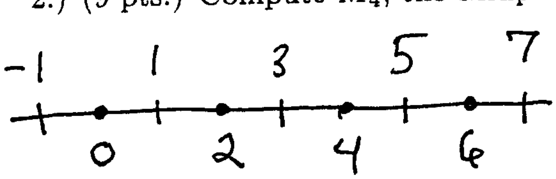
$\mu = e^{\int \frac{1}{x} dx} = e^{\ln|x| + \cancel{c}^{\,0}} = e^{\ln x} = x \rightarrow$

$xY' + Y = x^2 + \cos x \rightarrow D(xY) = x^2 + \cos x \rightarrow$

$xY = \int (x^2 + \cos x)\, dx \rightarrow$

$xY = \frac{1}{3}x^3 + \sin x + C$

2.) (9 pts.) Compute M_4, the Midpoint Estimate with $n = 4$, for $\int_{-1}^{7} \underbrace{\frac{5}{\sqrt{x^2+1}}}_{f(x)}\, dx$



$n = 4, \quad h = \frac{b-a}{n} = \frac{7-(-1)}{4} = 2$, then

$M_4 = h\left[f(0) + f(2) + f(4) + f(6)\right]$
$= 2\left[\frac{5}{\sqrt{1}} + \frac{5}{\sqrt{5}} + \frac{5}{\sqrt{17}} + \frac{5}{\sqrt{37}}\right] \approx 18.54$

3.) (6 pts. each) Determine the equilibria for $\frac{dN}{dt} = \overbrace{N^4 + N^3 - 2N^2}^{g(N)}$ and determine their stability using $= N^2(N^2+N-2) = N^2(N-1)(N+2) = 0$

a.) the graphical approach (sign chart method).

$+$	0	$-$	0	$-$	0	$+$	$g(N)$
	$N=-2$		$N=0$		$N=1$		
	(STABLE)		(SEMI-STABLE)		(UNSTABLE)		

b.) the analytical approach (eigenvalue method).

$$g(N) = N^4 + N^3 - 2N^2 \xrightarrow{D} g'(N) = 4N^3 + 3N^2 - 4N;$$

$$g'(-2) = 4(-2)^3 + 3(-2)^2 - 4(-2) = -12 < 0 \quad \text{(STABLE)} \ (N=-2)$$

$$g'(1) = 4(1)^3 + 3(1)^2 - 4(1) = 3 > 0 \quad \text{(UNSTABLE)} \ (N=1)$$

$$g'(0) = 0, \ g''(N) = 12N^2 + 6N - 4 \to g''(0) = -4 \ (\cap)$$

(SEMI-STABLE) $(N=0)$

4.) (8 pts.) What should n be in order that the Midpoint Estimate estimate the exact value of $\int_{-1}^{1} (x+2)^{-1}\, dx$ with absolute error at most 0.0001 ? The absolute error for the Midpoint Estimate is given by $|E_n| \le (b-a)\frac{h^2}{24}\left\{\max_{a \le x \le b} |f''(x)|\right\}$.

$$f(x) = (x+2)^{-1} \xrightarrow{D} f'(x) = -(x+2)^{-2} \xrightarrow{D} f''(x) = 2(x+2)^{-3};$$

$$\max_{-1 \le x \le 1} |f''(x)| = \max_{-1 \le x \le 1} \frac{2}{|x+2|^3} = \frac{2}{|(-1)+2|^3} = \frac{2}{1} = 2;$$

$h = \frac{1-(-1)}{n} = \frac{2}{n}$, then

$$|E_n| \le (1-(-1))\frac{\left(\frac{2}{n}\right)^2}{\cancel{24}\,12}\{\cancel{2}\}$$

$$= \frac{2}{12} \cdot \frac{4}{n^2}$$

$$= \frac{2}{3n^2} \le 0.0001$$

$$n^2 \ge \frac{2}{3(0.0001)} \to$$

$$n \ge \sqrt{\frac{2}{0.0003}} \approx 81.65$$

so choose

$$\boxed{n = 82}$$

5.) (8 pts. each) Evaluate the following improper integrals.

a.) $\displaystyle\int_1^\infty \frac{4x}{1+x^2}\,dx \quad = \lim_{A\to\infty} \int_1^A \frac{4x}{1+x^2}\,dx$

$$= \lim_{A\to\infty} 2\ln|1+x^2|\Big|_1^A$$

$$= \lim_{A\to\infty} \left(2\ln|1+\overset{\infty}{A^2}| - 2\ln 2\right)$$

$$= \infty - 2\ln 2$$

$$= \infty$$

b.) $\displaystyle\int_4^{13} \frac{1}{\sqrt{x-4}}\,dx \quad = \lim_{A\to 4^+} \int_A^{13} (x-4)^{-1/2}\,dx$

$$= \lim_{A\to 4^+} 2(x-4)^{1/2}\Big|_A^{13}$$

$$= \lim_{A\to 4^+} \left(2(9)^{1/2} - 2(A-4)^{1/2}\right)$$

$$= 2(3) - 2(0)^{1/2}$$

$$= 6 - 0$$

$$= 6$$

6.) (9 pts.) Determine $P_2(x)$, the second-degree Taylor Polynomial centered at $x = -1$, for $f(x) = x^2 + 8 \cdot \sqrt{x+2}$.

$\xrightarrow{D} f'(x) = 2x + 4(x+2)^{-1/2} \xrightarrow{D} f''(x) = 2 - 2(x+2)^{-3/2}$;

$a_0 = \dfrac{f(-1)}{0!} = \dfrac{9}{1} = 9$,

$a_1 = \dfrac{f'(-1)}{1!} = \dfrac{2}{1} = 2$,

$a_2 = \dfrac{f''(-1)}{2!} = 0$;

$P_2(x) = a_0 + a_1(x-(-1)) + a_2(x-(-1))^2$

$= 9 + 2(x+1) + (0)(x+1)^2$

$= 9 + 2(x+1)$

7.) (9 pts.) (Mixture Problem) Let S be the amount (pounds) of sugar in a tank at time t (minutes). A solution containing 3 pounds of sugar per gallon begins flowing into the tank at the rate of 4 gallons per minute and the well-stirred mixture flows out of the tank at the rate of 6 gallons per minute. Initially, the tank holds 300 gallons with 35 pounds of sugar. SET UP BUT DO NOT SOLVE a differential equation with initial conditions for the rate $\dfrac{dS}{dt}$.

$\dfrac{dS}{dt} =$ (RATE IN) − (RATE OUT)

$= \left(\dfrac{3 \text{ lbs.}}{\text{gal.}}\right)\left(\dfrac{4 \text{ gal.}}{\text{min.}}\right) - \left(\dfrac{S \text{ lbs.}}{300-2t \text{ gal.}}\right)\left(\dfrac{6 \text{ gal.}}{\text{min.}}\right)$

and $S(0) = 35$ lbs.

8.) (10 pts.) The following data are plotted on a semi-log graph on the following page. Use the graph (Assume it is a line.) to solve for N and determine the growth rate $\frac{dN}{dt}$ in autonomous form.

t	N
0	200
2	50
3	25
4	12.5
6	3.125

$Y = mt + b \rightarrow$

$\log N = mt + \log 200$

and $t = 2$, $N = 50$ so

$\log 50 = 2m + \log 200 \rightarrow$

$2m = \log 50 - \log 200 = \log \frac{50}{200} = \log \frac{1}{4}$

$\rightarrow m = \frac{1}{2} \log \frac{1}{4}$; then

$\log N = \left(\frac{1}{2} \log \frac{1}{4}\right) t + \log 200$ (Solve for N.)

$\rightarrow \log N = \frac{t}{2} \log \frac{1}{4} + \log 200$

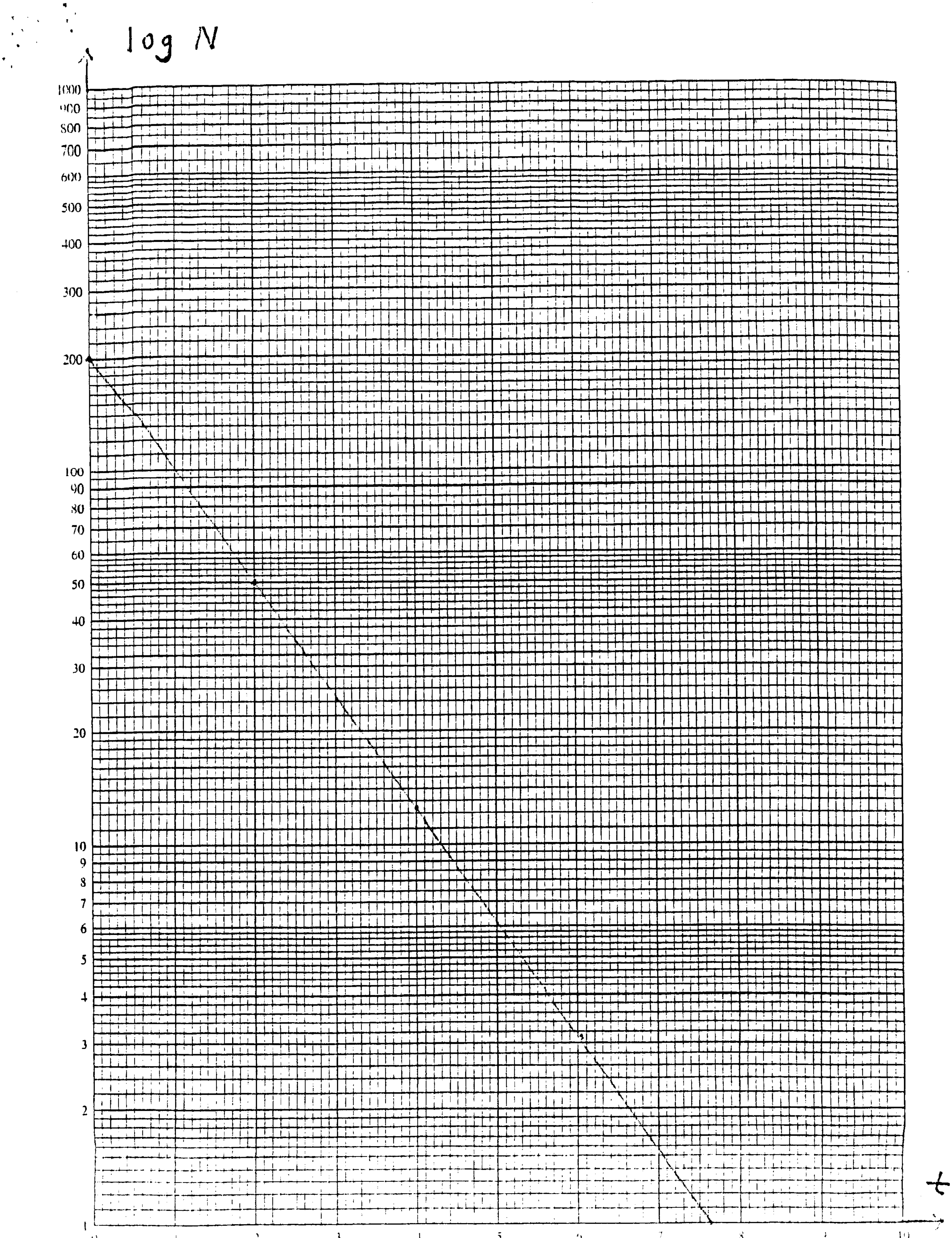
$\rightarrow \log N = \log \left(\frac{1}{4}\right)^{t/2} + \log 200$

$\rightarrow \log N = \log \left(200 \left(\frac{1}{4}\right)^{t/2}\right)$

$\rightarrow 10^{\log N} = 10^{\log \left(200 \left(\frac{1}{4}\right)^{t/2}\right)}$

$\rightarrow \boxed{N = 200 \left(\frac{1}{4}\right)^{t/2} = 200 \left(\frac{1}{2}\right)^{t}} \xrightarrow{D}$

$\frac{dN}{dt} = \underbrace{200 \cdot \left(\frac{1}{2}\right)^{t}} \cdot \ln\left(\frac{1}{2}\right) \rightarrow \boxed{\frac{dN}{dt} = N \cdot \ln\left(\frac{1}{2}\right)}$



log N
1000
900
800
700
600
500
400
300
200
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
t

The following EXTRA CREDIT problem is OPTIONAL. It is worth 8 points.

1.) Determine the following integral : $\int \sec^3 x \, dx$

$$\int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \, dx$$

(Let $u = \sec x$, $dv = \sec^2 x \, dx$

$\rightarrow du = \sec x \tan x$, $v = \tan x$)

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \cdot (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x| + c$$

$$\longrightarrow 2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + c$$

$$\rightarrow \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + c$$