Math 17B (Spring 2017)
Kouba
Exam 3

Your Name : KEY

Your EXAM ID Number ___________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You have until 11:02 a.m. sharp to finish the exam. Failure to stop working when time is called may lead to a deduction in your exam score. Thank you for your cooperation.

1.) Consider the Leslie matrix $L = \begin{pmatrix} 0.8 & 1.6 & 2.5 & 1.2 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{pmatrix}$

a.) (2 pts.) How many age classes are in this population ? 4

b.) (2 pts.) What percentage of 2-year old females survive to the end of the following breeding season ? 30 %

c.) (2 pts.) What percenatge of 3-year old females survive to the end of the following breeding season ? 0 %

d.) (2 pts.) What is an average number of female offspring for a 0-year old female ? 0.8

e.) (2 pts.) What is an average number of female offspring for a 1-year old female ? 1.6

f.) (6 pts.) If $N(0) = \begin{pmatrix} 20 \\ 30 \\ 10 \\ 40 \end{pmatrix}$, determine $N(1)$. How many 1-year old females will there be at the end of breeding season 1 ?

$$\begin{bmatrix} 0.8 & 1.6 & 2.5 & 1.2 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \\ 10 \\ 40 \end{bmatrix} = \begin{bmatrix} 16+48+25+48 \\ 14 \\ 12 \\ 30 \end{bmatrix} = \begin{bmatrix} 137 \\ 14 \\ 12 \\ 3 \end{bmatrix}$$

14 1-yr. olds

2.) (8 pts.) Regular M & M's cost \$4/lb. and Almond M & M's cost \$8/lb. How many pounds of each type of M & M's should be mixed together in order to result in a mixture weighing 20 pounds and costing \$5/lb. ? Use any method to solve the problem.

Let x: lbs. reg., y: lbs. alm.

$$\begin{cases} x+y=20 & \text{(lbs.)} \\ 4x+8y = 5(20) = 100 & (\$) \end{cases} \rightarrow$$

$y = 20-x \rightarrow$ (SUB) $\rightarrow 4x+8(20-x) = 100$

$\rightarrow 4x+160-8x = 100 \rightarrow 4x = 60 \rightarrow$

$x = 15$ lbs., $y = 5$ lbs.

3.) Use matrix reduction to solve each of the following systems of equations.

a.) (7 pts.) $\begin{cases} -3x + 2y = 7 \\ 4x + y = -2 \end{cases}$

$$\left[\begin{array}{cc|c} -3 & 2 & 7 \\ 4 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & 2 & 7 \\ 1 & 3 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 0 & 11 & 22 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & -1 \end{array}\right]$$

$$\rightarrow x = -1, \quad y = 2$$

(9 pts.)

b.) $\begin{cases} 2x + y - z = 2 \\ 3x - y + 2z = 1 \\ x - 2y + 3z = -1 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 5 & -7 & 4 \\ 0 & 5 & -7 & 4 \\ 1 & -2 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -7/5 & 4/5 \\ 1 & -2 & 3 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -7/5 & 4/5 \\ 1 & 0 & 1/5 & 3/5 \end{array}\right] \rightarrow \begin{cases} x + \frac{1}{5}z = 3/5 \\ y - \frac{7}{5}z = 4/5 \end{cases}, \text{ so}$$

let $\boxed{z = t}$ any # $\rightarrow$

$$\boxed{x = 3/5 - 1/5\, t},$$

$$\boxed{y = 4/5 + 7/5\, t}$$

4.) Determine the inverse for each matrix.

a.) (8 pts.) $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$

$$\left[\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & -2 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

$$\rightarrow \quad A^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \quad \text{OR}$$

$$\det A = (5)(3) - (7)(2) = 1 \rightarrow A^{-1} = \frac{1}{(1)} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

b.) (10 pts.) $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 2 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

5.) (10 pts.) The given 2x2 matrix is a Leslie matrix. Determine a stable age distribution for this matrix and the percentage of 1 -year olds in the stable age distribution. Is the population increasing or decreasing in size ?

$$L = \begin{pmatrix} 1 & 3 \\ 2/3 & 0 \end{pmatrix}$$

$$\det(L-\lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ 2/3 & 0-\lambda \end{bmatrix}$$

$$= (1-\lambda)(-\lambda) - (3)(2/3) = \lambda^2 - \lambda - 2$$

$$= (\lambda-2)(\lambda+1) = 0 \rightarrow \boxed{\lambda_1 = 2}, \quad \lambda_2 = -1;$$

$\boxed{\lambda_1 = 2}$: Solve $(L-\lambda I)X = 0$:

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 2/3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \quad -x_1 + 3x_2 = 0 \text{ so}$$

let $x_2 = t$ any # $\rightarrow$ $x_1 = 3x_2 = 3t$ $\rightarrow$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so let eigenvector is

$V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and stable age distribution

is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$; the % of 1-yr. olds is

$\frac{1}{1+3} = \frac{1}{4} = 25\%$; $\lambda_1 = 2 > 1$, so

population is $\uparrow$.

6.) (12 pts.) Find eigenvalues and the corresponding eigenvectors for the matrix
$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda) - (3)(4)$$

$$= \lambda^2 - 3\lambda + 2 - 12 = \lambda^2 - 3\lambda - 10$$

$$= (\lambda-5)(\lambda+2) = 0 \rightarrow \lambda = 5, \ \lambda = -2$$

For $\boxed{\lambda = 5}$: Solve $(A-\lambda I)X = 0 \rightarrow$

$$\left[\begin{array}{cc|c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$$

$-4x_1 + 3x_2 = 0$, so let $x_2 = t$ any # $\rightarrow$

$4x_1 = 3x_2 = 3t \rightarrow x_1 = \frac{3}{4}t \rightarrow$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = \frac{t}{4}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \rightarrow$ e.v. is $V_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$;

For $\boxed{\lambda = -2}$: Solve $(A-\lambda I)X = 0 \rightarrow$

$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 + x_2 = 0$, so

let $x_2 = t$ any # $\rightarrow x_1 = -x_2 = -t \rightarrow$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow$$

e.v. is $V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

7.) (7 pts.) Consider the given matrices of different sizes. Some of the following matrix operations make sense and some do not make sense. After each part circle YES if it makes sense and circle NO if it does not make sense.

$$A = \begin{pmatrix} 2 \\ -3 \end{pmatrix},\ B = \begin{pmatrix} 4 & 1 \end{pmatrix},\ C = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix},\ E = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix}$$

a.) $A + B$ YES NO

b.) CD YES NO

c.) AB YES NO

d.) D^{-1} YES NO

e.) $E - D$ YES NO

f.) BA YES NO

g.) $AB + C$ YES NO

8.) a.) (4 pts.) Give an example of a 2×2 rotation matrix which rotates vectors $\pi/3$ radians counter-clockwise.

$$R = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

b.) (3 pts.) Find the determinant of your matrix in part a.

$$\det R = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 1$$

9.) (6 pts.) If possible, find two 2×2 matrices A and B, both of which have at least one non-zero entry, so that $AB = -BA$. If not possible, briefly explain why.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -AB$$

The following EXTRA CREDIT problem is OPTIONAL. It is worth 8 points.

1.) We say that a $2x2$ matrix A is orthogonal if $AA' = I$, where A' is the transpose of A. Find two different examples of orthogonal matrices.

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a^2+b^2 & bc \\ bc & c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightarrow c^2 = 1 \rightarrow c = \pm 1$;

$\rightarrow bc = 0 \rightarrow b = 0$ or $\cancel{c = 0}$;

$\rightarrow a^2 = 1 \rightarrow a = \pm 1$; so

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ work}$$