Math 17C (Kouba)
Determining Stability of Equilibria for Nonlinear 2x2 Systems of D.E.'s Using the ANALYTICAL APPROACH

Ex: Find all equilibria for the following system of D.E.'s. Determine if they are STABLE or UNSTABLE, then categorize them.

$$\begin{cases} \dfrac{dx_1}{dt} = x_1^2 - x_2^2 = f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = x_1^2 - x_2 = f_2(x_1, x_2) \end{cases};$$

$x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = 0 \rightarrow$

$\boxed{x_1 = x_2}$ OR $\boxed{x_1 = -x_2}$; and

$x_1^2 - x_2 = 0 \rightarrow \boxed{x_2 = x_1^2}$; then

$\boxed{x_1 = x_2} \rightarrow (SOB) \rightarrow \; x_2 = (x_2)^2 \rightarrow$

$0 = x_2^2 - x_2 = x_2(x_2 - 1) \rightarrow$

$x_2 = 0$ OR $x_2 = 1$; $x_2 = 0 \rightarrow x_1 = 0$

so $\boxed{(0,0)}$ is equilibrium ;

$x_2 = 1 \rightarrow x_1 = 1$ so $\boxed{(1,1)}$ is equilibrium ; $\boxed{x_1 = -x_2} \rightarrow$

$(SOB) \rightarrow \; x_2 = (-x_2)^2 \rightarrow$

$0 = x_2^2 - x_2 = x_2(x_2 - 1) \rightarrow$

$x_2 = 0$ OR $x_2 = 1$; $x_2 = 0 \rightarrow x_1 = 0$;

$x_2 = 1 \rightarrow x_1 = -1$ so $\boxed{(-1,1)}$ is equilibrium.

The <u>JACOBI MATRIX</u> is

$$Df(x_1, x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_1 & -1 \end{bmatrix}$$

For (0,0): $Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = -\lambda(-1-\lambda) = 0$$

$\rightarrow \boxed{\lambda_1 = 0}$, $\lambda_2 = -1$ so NOT INTERESTING (BUT (0,0) UNSTABLE)

For (1,1): $Df(1,1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \rightarrow$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -2 \\ 2 & -1-\lambda \end{bmatrix}$$

$$= (2-\lambda)(-1-\lambda) - (-2)(2)$$

$$= -2 - 2\lambda + \lambda + \lambda^2 + 4$$

$$= \lambda^2 - \lambda + 2 = 0 \rightarrow \lambda = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$

$= \underbrace{\frac{1}{2}}_{(+)} \pm \frac{1}{2}\sqrt{7}\, i$ so (1,1) is UNSTABLE (UNSTABLE SPIRAL)

For $(-1,1)$: $Df(-1,1) = \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix} \rightarrow$

$$\det(A-\lambda I) = \det\begin{bmatrix} -2-\lambda & -2 \\ -2 & -1-\lambda \end{bmatrix}$$

$$= (-2-\lambda)(-1-\lambda) - (-2)(-2)$$

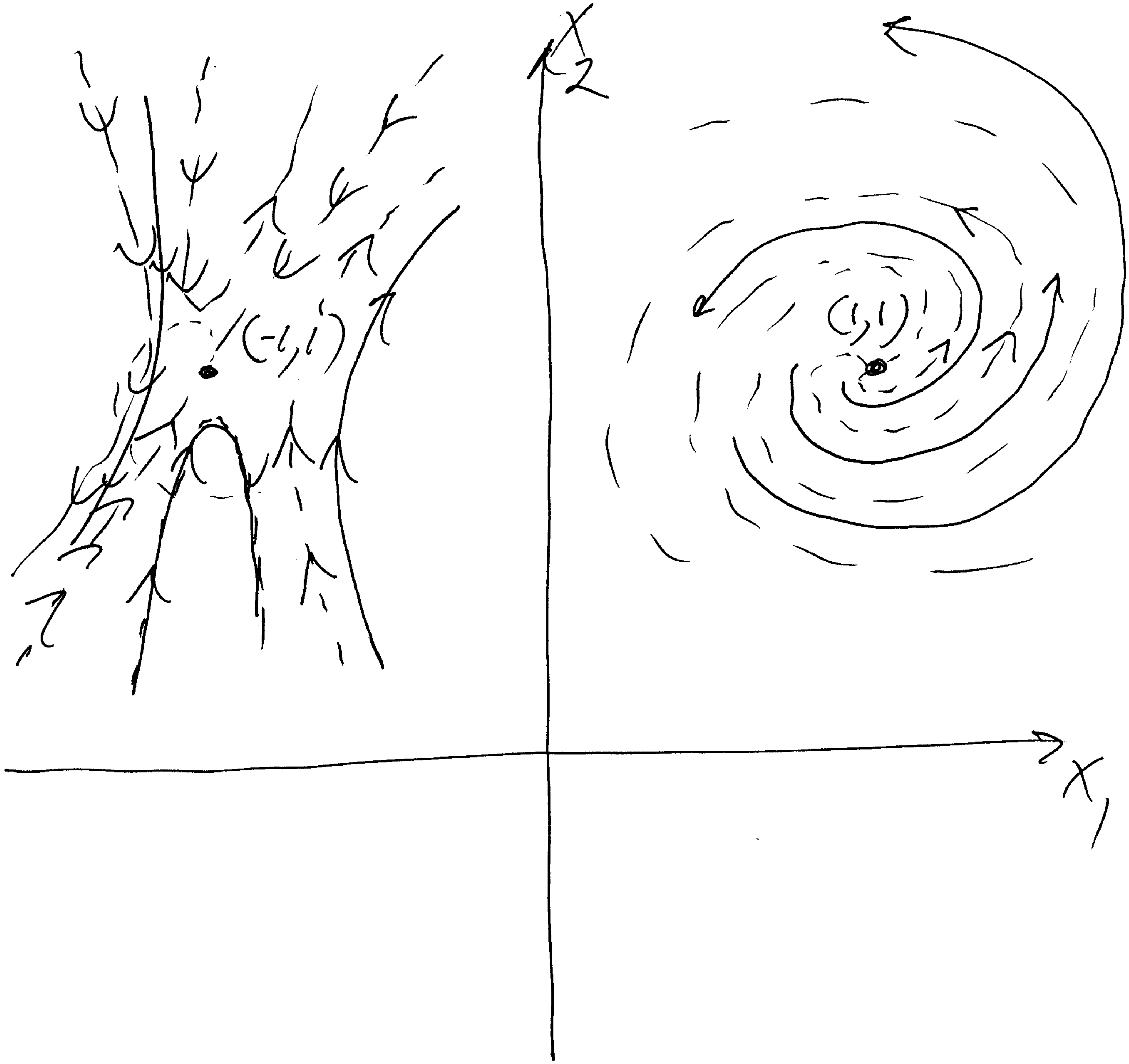
$$= 2 + 2\lambda + \lambda + \lambda^2 - 4$$

$$= \lambda^2 + 3\lambda - 2 = 0 \rightarrow$$

$$\lambda = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-2)}}{2(1)} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{17}$$

$$\rightarrow \lambda_1 = -\frac{3}{2} - \frac{1}{2}\sqrt{17} < 0, \quad \lambda_2 = -\frac{3}{2} + \frac{1}{2}\sqrt{17} \approx 0.56 > 0$$

so $(-1,1)$ UNSTABLE (SADDLE)



DIRECTION FIELD