Section 10.1 Problems

1. Cardiac output (CO) is a physiological quantity that is calculated as the product of heart rate (HR) and stroke volume (SV). Write cardiac output as a function of heart rate and stroke volume. If heart rate is measured in beats per minute and stroke volume in liters per beat, what is the unit for cardiac output? Determine the domain and range of the function describing cardiac output.

2. Mean arterial blood pressure (MAP) is a function of systolic blood pressure (SP) and diastolic blood pressure (DP). At a resting heart rate,

$$\text{MAP} \approx \text{DP} + \frac{1}{3}(\text{SP} - \text{DP})$$

If systolic pressure is greater than diastolic pressure and both are nonnegative, what is the range of the function describing mean arterial pressure?

3. Locate the following points in a three-dimensional Cartesian coordinate system:
(a) $(1, 3, 2)$ (b) $(-1, -2, 1)$
(c) $(0, 1, 2)$ (d) $(2, 0, 3)$

4. Describe in words the set of all points in $\mathbf{R}^3$ that satisfy the following expressions:
(a) $x = 0$ (b) $y = 0$ (c) $z = 0$
(d) $z \geq 0$ (e) $y \leq 0$

In Problems 5–12, evaluate each function at the given point.

5. $f(x, y) = \dfrac{2x}{x^2 + y^2}$ at $(2, 3)$

6. $f(x, y, z) = \sqrt{x^2 - 3y + z}$ at $(3, -1, 1)$

7. (a) $f_1(x, y) = 2x - 3y^2$ at $(-1, 2)$
(b) $f_2(y, x) = 2x - 3y^2$ at $(-1, 2)$

8. (a) $f_1(x, y) = \dfrac{x}{y}$ at $(3, 2)$ (b) $f_2(y, x) = \dfrac{x}{y}$ at $(3, 2)$
(c) $f_3(y, x) = \dfrac{y}{x}$ at $(3, 2)$

9. $h(x, t) = \exp\left[-\dfrac{(x-2)^2}{2t}\right]$ at $(1, 5)$

10. $g(n, p) - np(1 - p)^{n-1}$ at $(5, 0.1)$

11. $h(x_1, x_2) = x_2 e^{-x_1/x_2}$ at $(2, -1)$

12. $g(x_1, x_2, x_3, x_4) = x_1 x_4 \sqrt{x_2 x_3}$ at $(1, 8, 2, -1)$

In Problems 13–18, find the largest possible domain and the corresponding range of each function. Determine the equation of the level curves $f(x, y) = c$, together with the possible values of c.

13. $f(x, y) = x^2 + y^2$ 14. $f(x, y) = \sqrt{9 - x^2 - y^2}$

15. $f(x, y) = \ln(y - x^2)$ 16. $f(x, y) = \exp[-(x^2 + y^2)]$

17. $f(x, y) = \dfrac{x - y}{x + y}$ 18. $f(x, y) = \dfrac{x + y}{x - y}$

In Problems 19–22, match each function with the appropriate graph in Figures 10.21–10.24.

19. $f(x, y) = 1 + x^2 + y^2$ 20. $f(x, y) = \sin(x)\sin(y)$

21. $f(x, y) = y^2 - x^2$ 22. $f(x, y) = 4 - x^2$

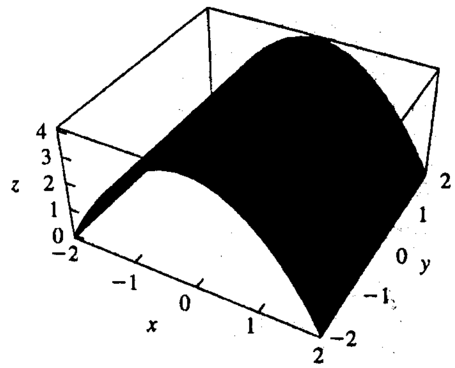

Figure 10.21

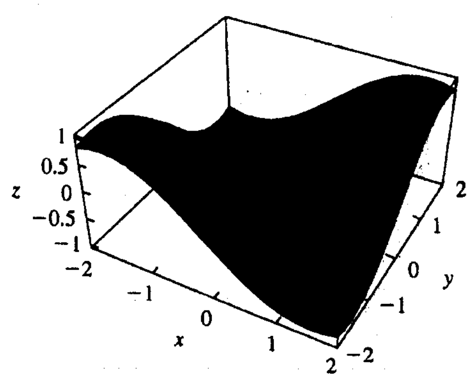

Figure 10.22

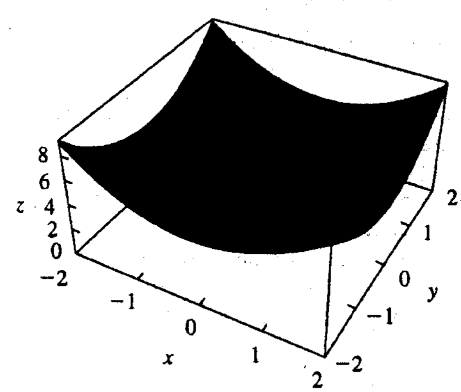

Figure 10.23

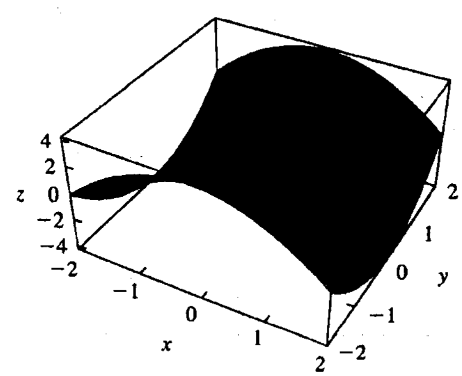

Figure 10.24

23. Let

$$f_a(x, y) = ax^2 + y^2$$

for $(x, y) \in \mathbf{R}$, where a is a positive constant.

(a) Assume that $a = 1$ and describe the level curves of f_1. The graph of $f_1(x, y)$ intersects both the x–z and the y–z planes; show that these two curves of intersection are parabolas.

(b) Assume that $a = 4$. Then

$$f_4(x, y) = 4x^2 + y^2$$

and the level curves satisfy

$$4x^2 + y^2 = c$$

Use a graphing calculator to sketch the level curves for $c = 0, 1, 2, 3,$ and 4. These curves are ellipses. Find the curves of intersection of $f_4(x, y)$ with the x–z and the y–z planes.

(c) Repeat (b) for $a = 1/4$.

(d) Explain in words how the surfaces of $f_a(x, y)$ change when a changes.

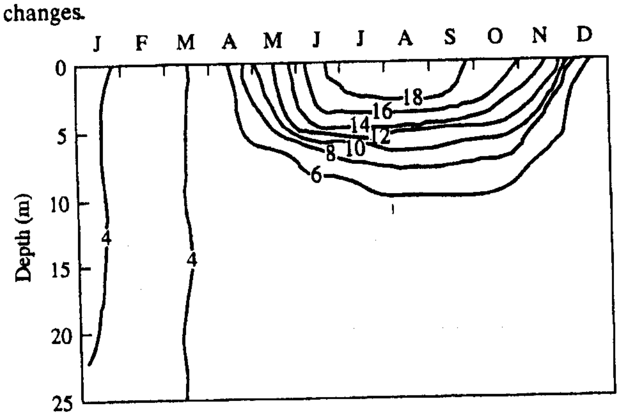

Figure 10.25 Isotherms for a typical lake in the Northern Hemisphere.

24. The graph in Figure 10.25 shows isotherms of a lake in the temperate climate of the Northern Hemisphere.

(a) Use this plot to sketch the temperature profiles in March and June. That is, plot the temperature as a function of depth for a day in March and for a day in June.

(b) Explain how it follows from your temperature plots that the lake is **homeothermic**—that is, has the same temperature from the surface to the bottom—in March.

(c) Explain how it follows from your temperature plots that the lake is **stratified**—that is, has a warm layer on top (called the **epilimnion**), followed by a region where the temperature changes quickly (called the **metalimnion**), followed by a cold layer deeper down (called the **hypolimnion**)—in June.

25. Figure 10.26 shows the oxygen concentration for Long Lake, Clear Water County (Minnesota). The water flea *Daphnia* can survive only if the oxygen concentration is higher than 3 mg/l. Suppose that you wanted to sample the *Daphnia* population in 1997 on days 180, 200, and 220. Below which depths can you be fairly sure not to find any *Daphnia*?

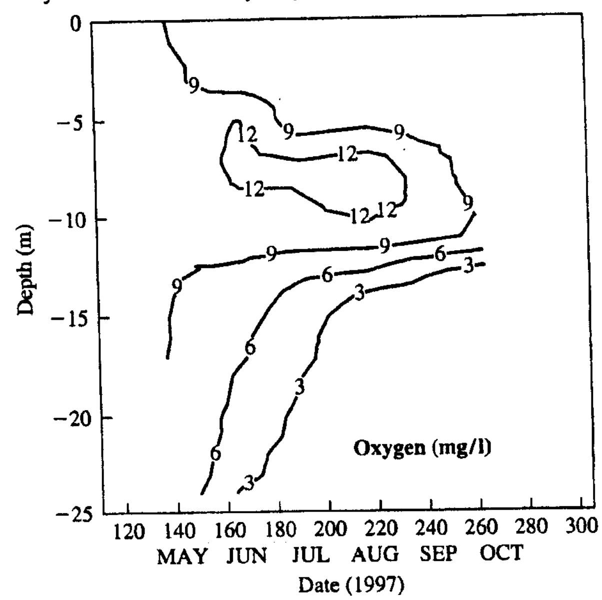

Figure 10.26 Level curves for oxygen concentration on Long Lake, Clear Water County (Minnesota). Courtesy of Leif Hembre.

26. At the beginning of this chapter, we discussed the minimum temperature required for survival as a function of metabolic heat production and whole-body thermal conductance. Suppose that you wish to go winter camping in Northern Minnesota and the predicted low temperature for the night is −15°F. Use the information provided at the beginning of the chapter to find the maximum value of g_{Hb} for your sleeping bag that would allow you to sleep safely.

10.2 Limits and Continuity

10.2.1 Informal Definition of Limits

We need to extend the notion of limits and continuity to the multivariable setting. The ideas are the same as in the one-dimensional case. We will discuss only the two-dimensional case, but note that everything in this section can be generalized to higher dimensions.

Let's start with an informal definition of limits. We say that the "limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) is equal to L" if $f(x, y)$ can be made arbitrarily close to L whenever the point (x, y) is sufficiently close (but not equal) to the point (x_0, y_0). We denote this concept by

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

But this is exactly what we need to show. In other words, we have shown that, for every $\epsilon > 0$, we can find a $\delta > 0$ (namely $\delta = \sqrt{\epsilon}$) such that whenever (x, y) is close to $(0, 0)$, it follows that $x^2 + y^2$ is close to 0. ■

Section 10.2 Problems

■ 10.2.1

In Problems 1–14, use the properties of limits to calculate the following limits:

1. $\lim_{(x,y)\to(1,0)}(x^2 - 3y^2)$
2. $\lim_{(x,y)\to(-1,1)}(2xy + 3x^2)$
3. $\lim_{(x,y)\to(2,-1)}(x^2y^3 - 3xy)$
4. $\lim_{(x,y)\to(1,-2)}(2x^3 - 3y)(xy - 2)$
5. $\lim_{(x,y)\to(-1,3)} x^2(y^2 - 3xy)$
6. $\lim_{(x,y)\to(-5,1)} y(xy + x^2y^2)$
7. $\lim_{(x,y)\to(0,2)}\left(4xy^2 - \frac{x+1}{y}\right)$
8. $\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2}$
9. $\lim_{(x,y)\to(1,0)} \frac{x^2+y^2}{x^2-y^2}$
10. $\lim_{(x,y)\to(-1,3)} \frac{x^2-xy}{2x+y}$
11. $\lim_{(x,y)\to(0,1)} \frac{2xy-3}{x^2+y^2+1}$
12. $\lim_{(x,y)\to(-1,-2)} \frac{x^2-y^2}{2xy+2}$
13. $\lim_{(x,y)\to(2,0)} \frac{2x+4y^2}{y^2+3x}$
14. $\lim_{(x,y)\to(1,-2)} \frac{2x^2+y}{2xy+3}$
15. Show that
$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$$
does not exist by computing the limit along the positive x-axis and the positive y-axis.
16. Show that
$$\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2}{x^2 + y^2}$$
does not exist by computing the limit along the positive x-axis and the positive y-axis.
17. Compute
$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2 + y^2}$$
along the x-axis, the y-axis, and the line $y = x$. What can you conclude?
18. Compute
$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^3}$$
along lines of the form $y = mx$, for $m \neq 0$. What can you conclude?
19. Compute
$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^3 + yx}$$
along lines of the form $y = mx$, for $m \neq 0$, and along the parabola $y = x^2$. What can you conclude?
20. Compute
$$\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^3 + y^6}$$
along lines of the form $y = mx$, for $m \neq 0$, and along the parabola $x = y^2$. What can you conclude?

■ 10.2.2

21. Use the definition of continuity to show that
$$f(x, y) = x^2 + y^2$$
is continuous at $(0, 0)$.
22. Use the definition of continuity to show that
$$f(x, y) = \sqrt{9 + x^2 + y^2}$$
is continuous at $(0, 0)$.
23. Show that
$$f(x, y) = \begin{cases} \frac{4xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$
is discontinuous at $(0, 0)$. (*Hint*: Use Problem 17.)
24. Show that
$$f(x, y) = \begin{cases} \frac{3xy}{x^2+y^3} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$
is discontinuous at $(0, 0)$. (*Hint*: Use Problem 18.)
25. Show that
$$f(x, y) = \begin{cases} \frac{2xy}{x^3+yx} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$
is discontinuous at $(0, 0)$. (*Hint*: Use Problem 19.)
26. Show that
$$f(x, y) = \begin{cases} \frac{3x^2y^2}{x^3+y^6} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$
is discontinuous at $(0, 0)$. (*Hint*: Use Problem 20.)
27. **(a)** Write
$$h(x, y) = \sin(x^2 + y^2)$$
as a composition of two functions.
(b) For which values of (x, y) is $h(x, y)$ continuous?
28. **(a)** Write
$$h(x, y) = \sqrt{x + y}$$
as a composition of two functions.
(b) For which values of (x, y) is $h(x, y)$ continuous?
29. **(a)** Write
$$h(x, y) = e^{xy}$$
as a composition of two functions.
(b) For which values of (x, y) is $h(x, y)$ continuous?
30. **(a)** Write
$$h(x, y) = \cos(y - x)$$
as a composition of two functions.
(b) For which values of (x, y) is $h(x, y)$ continuous?

■ 10.2.3

31. Draw an open disk with radius 2 centered at $(1, -1)$ in the x–y plane, and give a mathematical description of this set.

32. Draw a closed disk with radius 3 centered at (2, 0) in the x–y plane, and give a mathematical description of this set.

33. Give a geometric interpretation of the set

$$A = \left\{(x, y) \in \mathbf{R}^2 : \sqrt{x^2 + y^2 - 4y + 4} < 3\right\}$$

34. Give a geometric interpretation of the set

$$A = \left\{(x, y) \in \mathbf{R}^2 : \sqrt{x^2 + 6x + y^2 - 2y + 10} < 2\right\}$$

35. Let

$$f(x, y) = 2x^2 + y^2$$

Use the ϵ–δ definition of limits to show that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

36. Let

$$f(x, y) = x^2 + 3y^2$$

Use the ϵ–δ definition of limits to show that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

10.3 Partial Derivatives

10.3.1 Functions of Two Variables

Suppose that the response of an organism depends on a number of independent variables. To investigate this dependency, a common experimental design is to measure the response when one variable is changed while all other variables are kept fixed. As an example, Pisek et al. (1969) measured the net assimiliation of CO_2 of *Ranunculus glacialis*, a member of the buttercup family, as a function of temperature and light intensity. They varied the temperature while keeping the light intensity constant. Repeating this experiment at different light intensities, they were able to determine how the net assimiliation of CO_2 changes as a function of both temperature and light intensity.

This experimental design illustrates the idea behind partial derivatives. Suppose that we want to know how the function $f(x, y)$ changes when x and y change. Instead of changing both variables simultaneously, we might get an idea of how $f(x, y)$ depends on x and y when we change one variable while keeping the other variable fixed.

To illustrate, we look at

$$f(x, y) = x^2 y$$

We want to know how $f(x, y)$ changes if we change, say, x and keep y fixed. So we fix $y = y_0$. Then the change in f with respect to x is simply the derivative of f with respect to x when $y = y_0$. That is,

$$\frac{d}{dx} f(x, y_0) = \frac{d}{dx} x^2 y_0 = 2x y_0$$

Such a derivative is called a *partial derivative*.

Definition Suppose that f is a function of two independent variables x and y. The **partial derivative** of f with respect to x is defined by

$$\frac{\partial f(x, y)}{\partial x} = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

The partial derivative of f with respect to y is defined by

$$\frac{\partial f(x, y)}{\partial y} = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

To denote partial derivatives, we use "∂" instead of "d." We will also use the notation

$$f_x(x, y) = \frac{\partial f(x, y)}{\partial x} \quad \text{and} \quad f_y(x, y) = \frac{\partial f(x, y)}{\partial y}$$

The mixed-derivative theorem can be extended to higher-order derivatives. The order of differentiation does not matter, as long as the function and all of its derivatives through the order in question are continuous on an open disk centered at the point at which we want to compute the derivative. For instance,

$$\begin{aligned}\frac{\partial^3 f}{\partial y^2\,\partial x} &= \frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial x}(y^2 \sin x)\\ &= \frac{\partial}{\partial y}\frac{\partial}{\partial y}(y^2 \cos x)\\ &= \frac{\partial}{\partial y}(2y \cos x) = 2\cos x\end{aligned}$$

which is the same as $\partial^3 f/(\partial x\,\partial y^2)$.

Section 10.3 Problems

■ 10.3.1

In Problems 1–16, find $\partial f/\partial x$ and $\partial f/\partial y$ for the given functions.

1. $f(x,y) = x^2y + xy^2$
2. $f(x,y) = 2x\sqrt{y} - \frac{3}{xy^2}$
3. $f(x,y) = (xy)^{3/2} - (xy)^{2/3}$
4. $f(x,y) = \frac{y^4}{x^3} - \frac{1}{x^3y^4}$
5. $f(x,y) = \sin(x+y)$
6. $f(x,y) = \tan(x-2y)$
7. $f(x,y) = \cos^2(x^2-2y)$
8. $f(x,y) = \sec(y^2x - x^3)$
9. $f(x,y) = e^{\sqrt{x+y}}$
10. $f(x,y) = x^2e^{-xy/2}$
11. $f(x,y) = e^x\sin(xy)$
12. $f(x,y) = e^{-y^2}\cos(x^2-y^2)$
13. $f(x,y) = \ln(2x+y)$
14. $f(x,y) = \ln(3x^2-xy)$
15. $f(x,y) = \log_3(y^2-x^2)$
16. $f(x,y) = \log_5(3xy)$

In Problems 17–24, find the indicated partial derivatives.

17. $f(x,y) = 3x^2 - y - 2y^2$; $f_x(1,0)$
18. $f(x,y) = x^{1/3}y - xy^{1/3}$; $f_y(1,1)$
19. $g(x,y) = e^{x+3y}$; $g_y(2,1)$
20. $h(u,v) = e^u\sin(u+v)$; $h_u(1,-1)$
21. $f(x,z) = \ln(xz)$; $f_z(e,1)$
22. $g(v,w) = \frac{w^2}{v+w}$; $g_v(1,1)$
23. $f(x,y) = \frac{xy}{x^2+2}$; $f_x(-1,2)$
24. $f(u,v) = e^{u^2/2}\ln(u+v)$; $f_u(2,1)$
25. Let

$$f(x,y) = 4 - x^2 - y^2$$

Compute $f_x(1,1)$ and $f_y(1,1)$, and interpret these partial derivatives geometrically.

26. Let

$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

Compute $f_x(1,1)$ and $f_y(1,1)$, and interpret these partial derivatives geometrically.

27. Let

$$f(x,y) = 1 + x^2y$$

Compute $f_x(-2,1)$ and $f_y(-2,1)$, and interpret these partial derivatives geometrically.

28. Let

$$f(x,y) = 2x^3 - 3yx$$

Compute $f_x(1,2)$ and $f_y(1,2)$, and interpret these partial derivatives geometrically.

29. In Example 4, we investigated Holling's disk equation

$$P_e = \frac{aNT}{1 + aT_hN}$$

(See Example 4 for the meaning of this equation.) We will now consider P_e as a function of the predator attack rate a and the length T of the interval during which the predator searches for food.

(a) Determine how the predator attack rate a influences the number of prey eaten per predator.

(b) Determine how the length T of the interval influences the number of prey eaten per predator.

30. Suppose that the per capita growth rate of some prey at time t depends on both the prey density $H(t)$ at t and the predator density $P(t)$ at t. Assume the relationship

$$\frac{1}{H}\frac{dH}{dt} = r\left(1 - \frac{H}{K}\right) - aP \qquad (10.2)$$

where r, K, and a are positive constants. The right-hand side of (10.2) is a function of both prey density and predator density. Investigate how an increase in (a) prey density and (b) predator density affects the per capita growth rate of this prey species.

■ 10.3.2

In Problems 31–38, find $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ for the given functions.

31. $f(x,y,z) = x^2z + yz^2 - xy$
32. $f(x,y,z) = xyz$
33. $f(x,y,z) = x^3y^2z + \frac{x}{yz}$
34. $f(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$
35. $f(x,y,z) = e^{x+y+z}$
36. $f(x,y,z) = e^{yz}\sin x$
37. $f(x,y,z) = \ln(x+y+z)$
38. $f(x,y,z) = y\tan(x^2+z)$

■ 10.3.3

In Problems 39–48, find the indicated partial derivatives.

39. $f(x,y) = x^2y + xy^2$; $\frac{\partial^2 f}{\partial x^2}$
40. $f(x,y) = y^2(x-3y)$; $\frac{\partial^2 f}{\partial y^2}$
41. $f(x,y) = xe^y$; $\frac{\partial^2 f}{\partial x\partial y}$
42. $f(x,y) = \sin(x-y)$; $\frac{\partial^2 f}{\partial y\partial x}$
43. $f(u,w) = \tan(u+w)$; $\frac{\partial^2 f}{\partial u^2}$
44. $g(s,t) = \ln(s^2+3st)$; $\frac{\partial^2 g}{\partial t^2}$
45. $f(x,y) = x^3\cos y$; $\frac{\partial^3 f}{\partial x^2\partial y}$
46. $f(x,y) = e^{x^2-y}$; $\frac{\partial^3 f}{\partial y^2\partial x}$
47. $f(x,y) = \ln(x+y)$; $\frac{\partial^3 f}{\partial x^3}$
48. $f(x,y) = \sin(3xy)$; $\frac{\partial^3 f}{\partial y^2\partial x}$

49. The functional responses of some predators are sigmoidal; that is, the number of prey attacked per predator as a function of prey density has a sigmoidal shape. If we denote the prey density by N, the predator density by P, the time available for searching for prey by T, and the handling time of each prey item per predator by T_h, then the number of prey encounters per predator as a function of N, T, and T_h can be expressed as

$$f(N, T, T_h) = \frac{b^2 N^2 T}{1 + cN + bT_h N^2}$$

where b and c are positive constants.

(a) Investigate how an increase in the prey density N affects the function $f(N, T, T_h)$.

(b) Investigate how an increase in the time T available for search affects the function $f(N, T, T_h)$.

(c) Investigate how an increase in the handling time T_h affects the function $f(N, T, T_h)$.

(d) Graph $f(N, T, T_h)$ as a function of N when $T = 2.4$ hours, $T_h = 0.2$ hours, $b = 0.8$, and $c = 0.5$.

50. In this problem, we will investigate how mutual interference of parasitoids affects their searching efficiency for a host. We assume that N is the host density and P is the parasitoid density. A frequently used model for host–parasitoid interactions is the **Nicholson–Bailey model** (Nicholson, 1933; Nicholson and Bailey, 1935), in which it is assumed that the number of parasitized hosts, denoted by N_a, is given by

$$N_a = N[1 - e^{-bP}] \qquad (10.3)$$

where b is the searching efficiency.

(a) Show that

$$b = \frac{1}{P} \ln \frac{N}{N - N_a}$$

by solving (10.3) for b.

(b) Consider

$$b = f(P, N, N_a) = \frac{1}{P} \ln \frac{N}{N - N_a}$$

as a function of P, N, and N_a. How is the searching efficiency b affected when the parasitoid density increases?

(c) Assume now that the fraction of parasitized host depends on the host density; that is, assume that

$$N_a = g(N)$$

where $g(N)$ is a nonnegative, differentiable function. The searching efficiency b can then be written as follows as a function of P and N:

$$b = h(P, N) = \frac{1}{P} \ln \frac{N}{N - g(N)}$$

How does the searching efficiency depend on host density when $g(N)$ is a decreasing function of N? (Use the fact that $g(N) < N$.)

51. Leopold and Kriedemann (1975) measured the crop growth rate of sunflowers as a function of leaf area index and percent of full sunlight. (Leaf area index is the ratio of leaf surface area to the ground area the plant covers.) They found that, for a fixed level of sunlight, crop growth rate first increases and then decreases as a function of leaf area index. For a given leaf area index, the crop growth rate increases with the level of sunlight. The leaf area index that maximizes the crop growth rate is an increasing function of sunlight. Sketch the crop growth rate as a function of leaf area index for different values of percent of full sunlight.

■ 10.4 Tangent Planes, Differentiability, and Linearization

■ 10.4.1 Functions of Two Variables

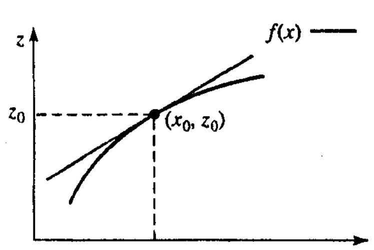

Figure 10.36 The curve $z = f(x)$ and its tangent line at the point (x_0, z_0).

Tangent Planes Suppose that $z = f(x)$ is differentiable at $x = x_0$. Then the equation of the tangent line of $z = f(x)$ at (x_0, z_0) with $z_0 = f(x_0)$ is given by

$$z - z_0 = f'(x_0)(x - x_0) \qquad (10.4)$$

The curve z and the tangent line are illustrated in Figure 10.36.

We now generalize this situation to functions of two variables. The analogue of a tangent line is called a **tangent plane**, an example of which is shown in Figure 10.37. Let $z = f(x, y)$ be a function of two variables. We saw in the previous section that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, evaluated at (x_0, y_0), are the slopes of tangent lines at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$, to certain curves through (x_0, y_0, z_0) on the surface $z = f(x, y)$. These two tangent lines, one in the x-direction, the other in the y-direction, define a unique plane. If, in addition, $f(x, y)$ has partial derivatives that are continuous on an open disk containing (x_0, y_0), then we can show that the tangent line at (x_0, y_0, z_0) to any other smooth curve on the surface $z = f(x, y)$ through (x_0, y_0, z_0) is contained in this plane. The plane is then called the *tangent plane*.

We will use the two original tangent lines to find the equation of the tangent plane at a point (x_0, y_0, z_0) on the surface $z = f(x, y)$. We take the curve that is obtained as the intersection of the surface $z = f(x, y)$ with the plane that is parallel to the y–z plane and contains the point (x_0, y_0, z_0)—that is, the plane $x = x_0$—and we denote this curve by C_1. Its tangent line at (x_0, y_0, z_0) is contained in the tangent plane. (See Figure 10.37.) Likewise, we take the curve of intersection between $z = f(x, y)$ and

The linearization of **f** about the point $(x_1^*, x_2^*, \ldots, x_n^*)$ is then

$$L(x_1^*, \ldots, x_n^*) = \begin{bmatrix} f_1(x_1^*, \ldots, x_n^*) \\ f_2(x_1^*, \ldots, x_n^*) \\ \vdots \\ f_m(x_1^*, \ldots, x_n^*) \end{bmatrix} + D\mathbf{f}(x_1^*, \ldots, x_n^*) \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}$$

Section 10.4 Problems

■ 10.4.1

In Problems 1–10, the tangent plane at the indicated point (x_0, y_0, z_0) exists. Find its equation.

1. $f(x, y) = 2x^3 + y^2$; $(1, 2, 6)$
2. $f(x, y) = x^2 - 3y^2$; $(-1, 1, -2)$
3. $f(x, y) = xy$; $(-1, -2, 2)$
4. $f(x, y) = \sin x + \cos y$; $(0, 0, 1)$
5. $f(x, y) = \sin(xy)$; $(1, 0, 0)$
6. $f(x, y) = e^{x-y}$; $(1, -1, e^2)$
7. $f(x, y) = e^{x^2+y^2}$; $(1, 0, e)$
8. $f(x, y) = e^x \cos y$; $(0, 0, 1)$
9. $f(x, y) = \ln(x + y)$; $(2, -1, 0)$
10. $f(x, y) = \ln(x^2 + y^2)$; $(1, 1, \ln 2)$

'n Problems 11–16, show that $f(x, y)$ is differentiable at the 'ndicated point.

11. $f(x, y) = y^2x + x^2y$; $(1, 1)$
12. $f(x, y) = xy - 3x^2$; $(1, 1)$
13. $f(x, y) = \cos(x + y)$; $(0, 0)$
14. $f(x, y) = e^{x-y}$; $(0, 0)$
15. $f(x, y) = x + y^2 - 2xy$; $(-1, 2)$
16. $f(x, y) = \tan(x^2 + y^2)$; $\left(\frac{\pi}{4}, -\frac{\pi}{4}\right)$

n Problems 17–24, find the linearization of $f(x, y)$ at the indicated 'oint (x_0, y_0).

7. $f(x, y) = x - 3y$; $(3, 1)$
8. $f(x, y) = 2xy$; $(1, -1)$
9. $f(x, y) = \sqrt{x} + 2y$; $(1, 0)$
0. $f(x, y) = \cos(x^2y)$; $\left(\frac{\pi}{2}, 0\right)$
1. $f(x, y) = \tan(x + y)$; $(0, 0)$
2. $f(x, y) = e^{3x+2y}$; $(1, 2)$
3. $f(x, y) = \ln(x^2 + y)$; $(1, 1)$
4. $f(x, y) = x^2e^y$; $(1, 0)$
5. Find the linear approximation of

$$f(x, y) = e^{x+y}$$

(0, 0), and use it to approximate $f(0.1, 0.05)$. Using a calculator, ɪmpare the approximation with the exact value of $f(0.1, 0.05)$.

ı. Find the linear approximation of

$$f(x, y) = \sin(x + 2y)$$

(0, 0), and use it to approximate $f(-0.1, 0.2)$. Using a lculator, compare the approximation with the exact value of $(-0.1, 0.2)$.

27. Find the linear approximation of

$$f(x, y) = \ln(x^2 - 3y)$$

at $(1, 0)$, and use it to approximate $f(1.1, 0.1)$. Using a calculator, compare the approximation with the exact value of $f(1.1, 0.1)$.

28. Find the linear approximation of

$$f(x, y) = \tan(2x - 3y^2)$$

at $(0, 0)$, and use it to approximate $f(0.03, 0.05)$. Using a calculator, compare the approximation with the exact value of $f(0.03, 0.05)$.

■ 10.4.2

In Problems 29–36, find the Jacobi matrix for each given function.

29. $\mathbf{f}(x, y) = \begin{bmatrix} x + y \\ x^2 - y^2 \end{bmatrix}$
30. $\mathbf{f}(x, y) = \begin{bmatrix} 2x - 3y \\ 4x^2 \end{bmatrix}$
31. $\mathbf{f}(x, y) = \begin{bmatrix} e^{x-y} \\ e^{x+y} \end{bmatrix}$
32. $\mathbf{f}(x, y) = \begin{bmatrix} (x - y)^2 \\ \sin(x - y) \end{bmatrix}$
33. $\mathbf{f}(x, y) = \begin{bmatrix} \cos(x - y) \\ \cos(x + y) \end{bmatrix}$
34. $\mathbf{f}(x, y) = \begin{bmatrix} \ln(x + y) \\ e^{x+y} \end{bmatrix}$
35. $\mathbf{f}(x, y) = \begin{bmatrix} 2x^2y - 3y + x \\ e^x \sin y \end{bmatrix}$
36. $\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ e^{-x^2} \end{bmatrix}$

In Problems 37–42, find a linear approximation to each function $f(x, y)$ at the indicated point.

37. $\mathbf{f}(x, y) = \begin{bmatrix} 2x^2y \\ \frac{1}{xy} \end{bmatrix}$ at $(1, 1)$
38. $\mathbf{f}(x, y) = \begin{bmatrix} 3x - y^2 \\ 4y \end{bmatrix}$ at $(-1, -2)$
39. $\mathbf{f}(x, y) = \begin{bmatrix} e^{2x-y} \\ \ln(2x - y) \end{bmatrix}$ at $(1, 1)$
40. $\mathbf{f}(x, y) = \begin{bmatrix} e^x \sin y \\ e^{-y} \cos x \end{bmatrix}$ at $(0, 0)$
41. $\mathbf{f}(x, y) = \begin{bmatrix} \frac{x}{y} \\ \frac{y}{x} \end{bmatrix}$ at $(1, 1)$
42. $\mathbf{f}(x, y) = \begin{bmatrix} (x + y)^2 \\ xy \end{bmatrix}$ at $(-1, 1)$
43. Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} x^2 - xy \\ 3y^2 - 1 \end{bmatrix}$$

at $(1, 2)$. Use your result to find an approximation for $f(1.1, 1.9)$, and compare the approximation with the value of $f(1.1, 1.9)$ that you get when you use a calculator.

44. Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} x/y \\ 2xy \end{bmatrix}$$

at $(-1, 1)$. Use your result to find an approximation for $f(-0.9, 1.05)$, and compare the approximation with the value of $f(-0.9, 1.05)$ that you get when you use a calculator.

45. Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} (x-y)^2 \\ 2x^2y \end{bmatrix}$$

at $(2, -3)$. Use your result to find an approximation for $f(1.9, -3.1)$, and compare the approximation with the value of $f(1.9, -3.1)$ that you get when you use a calculator.

46. Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{2x+y} \\ x - y^2 \end{bmatrix}$$

at $(1, 2)$. Use your result to find an approximation for $f(1.05, 2.05)$, and compare the approximation with the value of $f(1.05, 2.05)$ that you get when you use a calculator.

■ 10.5 More about Derivatives (Optional)

■ 10.5.1 The Chain Rule for Functions of Two Variables

In Section 10.3, we discussed how the net assimilation of CO_2 can change as a function of both temperature and light intensity. If we follow the net assimilation of CO_2 over time, we must take into account the fact that both temperature and light intensity depend on time. If we denote the temperature at time t by $T(t)$, the light intensity at time t by $I(t)$, and the net assimilation of CO_2 at time t by $N(t)$, then $N(t)$ is a function of both $T(t)$ and $I(t)$, and we can write

$$N(t) = f(T(t), I(t))$$

Net assimilation is thus a composite function.

To differentiate composite functions of one variable, we use the chain rule. Suppose that $w = f(x)$ is a function of one variable and that x depends on t. Then, by the chain rule, to differentiate w with respect to t, we have

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} \tag{10.12}$$

The chain rule can be extended to functions of more than one variable:

> **Chain Rule for Functions of Two Independent Variables** If $w = f(x, y)$ is differentiable and x and y are differentiable functions of t, then w is a differentiable function of t and
>
> $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

We will not prove this formula, but merely outline the steps that lead to it. We approximate $w = f(x, y)$ at (x_0, y_0) by its linear approximation

$$L(x, y) = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$

If we set $\Delta x = x - x_0$, $\Delta y = y - y_0$, and $\Delta w = f(x, y) - f(x_0, y_0)$, we can approximate $\Delta w = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ by its linear approximation. We find that

$$\Delta w \approx \frac{\partial f(x_0, y_0)}{\partial x}\Delta x + \frac{\partial f(x_0, y_0)}{\partial y}\Delta y$$

Dividing both sides by Δt, we obtain

$$\frac{\Delta w}{\Delta t} \approx \frac{\partial f(x_0, y_0)}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f(x_0, y_0)}{\partial y}\frac{\Delta y}{\Delta t}$$

Solution The gradient of f at $(1, 2)$ is perpendicular to the level curve at $(1, 2)$. The gradient of f is given as

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

Hence,

$$\nabla f(1, 2) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

To normalize this vector, we divide $\nabla f(1, 2)$ by its length. Since

$$|\nabla f(1, 2)| = \sqrt{(2)^2 + (-4)^2} = \sqrt{4 + 16} = 2\sqrt{5}$$

the unit vector that is perpendicular to the level curve of $f(x, y)$ at $(1, 2)$ is

$$\mathbf{u} = \frac{1}{2\sqrt{5}}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\sqrt{5} \\ -\frac{2}{5}\sqrt{5} \end{bmatrix}$$ ■

Section 10.5 Problems

■ 10.5.1

1. Let $f(x, y) = x^2 + y^2$ with $x(t) = 3t$ and $y(t) = e^t$. Find the derivative of $w = f(x, y)$ with respect to t when $t = \ln 2$.

2. Let $f(x, y) = e^x \sin y$ with $x(t) = t$ and $y(t) = t^3$. Find the derivative of $w = f(x, y)$ with respect to t when $t = 1$.

3. Let $f(x, y) = \sqrt{x^2 + y^2}$ with $x(t) = t$ and $y(t) = \sin t$. Find the derivative of $w = f(x, y)$ with respect to t when $t = \pi/3$.

4. Let $f(x, y) = \ln(xy - x^2)$ with $x(t) = t^2$ and $y(t) = t$. Find the derivative of $w = f(x, y)$ with respect to t when $t = 5$.

5. Let $f(x, y) = \frac{1}{x} + \frac{1}{y}$ with $x(t) = \sin t$ and $y(t) = \cos t$. Find the derivative of $w = f(x, y)$ with respect to t when $t = \pi/4$.

6. Let $f(x, y) = xe^y$ with $x(t) = e^t$ and $y(t) = t^2$. Find the derivative of $w = f(x, y)$ with respect to t when $t = 0$.

7. Find $\frac{dz}{dt}$ for $z = f(x, y)$ with $x = u(t)$ and $y = v(t)$.

8. Find $\frac{dw}{dt}$ for $w = e^{f(x,y)}$ with $x = u(t)$ and $y = v(t)$.

■ 10.5.2

9. Find $\frac{dy}{dx}$ if $(x^2 + y^2)e^y = 0$.

10. Find $\frac{dy}{dx}$ if $(\sin x + \cos y)x^2 = 0$.

11. Find $\frac{dy}{dx}$ if $\ln(x^2 + y^2) = 3xy$.

12. Find $\frac{dy}{dx}$ if $\cos(x^2 + y^2) = \sin(x^2 - y^2)$.

13. Find $\frac{dy}{dx}$ if $y = \arccos x$.

14. Find $\frac{dy}{dx}$ if $y = \arctan x$.

15. The growth rate r of a particular organism is affected by both the availability of food and the number of competitors for the food source. Denote the amount of food available at time t by $F(t)$ and the number of competitors at time t by $N(t)$. The growth rate r can then be thought of as a function of the two time-dependent variables $F(t)$ and $N(t)$. Assume that the growth rate is an increasing function of the availability of food and a decreasing function of the number of competitors. How is the growth rate r affected if the availability of food decreases over time while the number of competitors increases?

16. Suppose that you travel along an environmental gradient, along which both temperature and precipitation increase. If the abundance of a particular plant species increases with both temperature and precipitation, would you expect to encounter this species more often or less often during your journey? (Use calculus to answer this question.)

■ 10.5.3

In Problems 17–24, find the gradient of each function.

17. $f(x, y) = x^3 y^2$

18. $f(x, y) = \frac{xy}{x^2 + y^2}$

19. $f(x, y) = \sqrt{x^3 - 3xy}$

20. $f(x, y) = x(x^2 - y^2)^{2/3}$

21. $f(x, y) = \exp\left[\sqrt{x^2 + y^2}\right]$

22. $f(x, y) = \tan \frac{x-y}{x+y}$

23. $f(x, y) = \ln\left(\frac{x}{y} + \frac{y}{x}\right)$

24. $f(x, y) = \cos(3x^2 - 2y^2)$

In Problems 25–30, compute the directional derivative of $f(x, y)$ at the given point in the indicated direction.

25. $f(x, y) = \sqrt{2x^2 + y^2}$ at $(1, 2)$ in the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

26. $f(x, y) = x^2 \sin y$ at $(-1, 0)$ in the direction $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

27. $f(x, y) = e^{x+y}$ at $(0, 0)$ in the direction $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

28. $f(x, y) = x^3 y^2$ at $(2, 3)$ in the direction $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

29. $f(x, y) = 2xy^3 - 3x^2 y$ at $(1, -1)$ in the direction $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

30. $f(x, y) = ye^{x^2}$ at $(0, 2)$ in the direction $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

In Problems 31–34, compute the directional derivative of $f(x, y)$ at the point P in the direction of the point Q.

31. $f(x, y) = 2x^2 y - 3x$, $P = (2, 1)$, $Q = (3, 2)$

32. $f(x, y) = 4xy + y^2$, $P = (-1, 1)$, $Q = (3, 2)$

33. $f(x, y) = \sqrt{xy - 2x^2}$, $P = (1, 6)$, $Q = (3, 1)$

34. $f(x, y) = e^{x-y}$, $P = (2, 2)$, $Q = (1, -1)$

35. In what direction does $f(x, y) = 3xy - x^2$ increase most rapidly at $(-1, 1)$?

36. In what direction does $f(x, y) = e^x \cos y$ increase most rapidly at $(0, \pi/2)$?

37. In what direction does $f(x, y) = \sqrt{x^2 - y^2}$ increase most rapidly at $(5, 3)$?

38. In what direction does $f(x, y) = \ln(x^2 + y^2)$ increase most rapidly at $(1, 1)$?

39. Find a unit vector that is normal to the level curve of the function
$$f(x, y) = 3x + 4y$$
at the point $(-1, 1)$.

40. Find a unit vector that is normal to the level curve of the function
$$f(x, y) = x^2 + \frac{y^2}{9}$$
at the point $(1, 3)$.

41. Find a unit vector that is normal to the level curve of the function
$$f(x, y) = x^2 - y^3$$
at the point $(1, 3)$.

42. Find a unit vector that is normal to the level curve of the function
$$f(x, y) = xy$$
at the point $(2, 3)$.

43. **Chemotaxis** Chemotaxis is the chemically directed movement of organisms up a concentration gradient—that is, in the direction in which the concentration increases most rapidly. The slime mold *Dictyostelium discoideum* exhibits this phenomenon. Single-celled amoebas of this species move up the concentration gradient of a chemical called cyclic adenosine monophosphate (AMP). Suppose the concentration of cyclic AMP at the point (x, y) in the x–y plane is given by
$$f(x, y) = \frac{4}{|x| + |y| + 1}$$
If you place an amoeba at the point $(3, 1)$ in the x–y plane, determine in which direction the amoeba will move if its movement is directed by chemotaxis.

44. Suppose an organism moves down a sloped surface along the steepest line of descent. If the surface is given by
$$f(x, y) = x^2 - y^2$$
find the direction in which the organism will move at the point $(2, 3)$.

■ 10.6 Applications (Optional)

■ 10.6.1 Maxima and Minima

In Section 5.1, we introduced local extrema for functions of one variable. Local extrema can also be defined for functions of more than one independent variable; here, we will restrict our discussion to functions of two variables. Recall that we denoted by $B_\delta(x_0, y_0)$ the open disk with radius δ centered at (x_0, y_0). The following definition, with which you should compare the corresponding definition in Section 5.1, extends the notion of local extrema to functions of two variables:

Definition A function $f(x, y)$ defined on a set $D \subset \mathbf{R}^2$ has a **local (or relative) maximum** at a point (x_0, y_0) if there exists a $\delta > 0$ such that
$$f(x, y) \leq f(x_0, y_0) \quad \text{for all } (x, y) \in B_\delta(x_0, y_0) \cap D$$
A function $f(x, y)$ defined on a set $D \subset \mathbf{R}^2$ has a **local (or relative) minimum** at a point (x_0, y_0) if there exists a $\delta > 0$ such that
$$f(x, y) \geq f(x_0, y_0) \quad \text{for all } (x, y) \in B_\delta(x_0, y_0) \cap D$$

Informally, a local maximum (local minimum) is a point that is higher (lower) than all nearby points. We can define **global** (or **absolute**) extrema as well: If the inequalities in the definition hold for all $(x, y) \in D$, then f has an absolute maximum (minimum) at (x_0, y_0). Figure 10.43 shows an example of a function of two variables with a local maximum at $(0, 0)$.

How can we find local extrema? Recall that in the single-variable case, a horizontal tangent line at a point on the graph of a differentiable function is a necessary condition for the point to be a local extremum (Fermat's theorem). We can generalize this statement to functions of more than one variable: Looking at Figure 10.43, we see that the tangent plane at the local extremum is horizontal. The equation of a horizontal tangent plane on the graph of a differentiable function $f(x, y)$ at (x_0, y_0) is
$$z = f(x_0, y_0)$$
Comparing this equation with the general form of a tangent plane (Section 10.4), we

constant of $10^{-5}\,\text{cm}^2/\text{s}$, which means that it takes an oxygen molecule roughly 500 seconds to cross a distance of 1 mm by diffusion alone. Ribonuclease (an enzyme that hydrolyzes ribonucleic acid) in water at 20°C has a diffusion constant of $1.1 \times 10^{-6}\,\text{cm}^2/\text{s}$, which means that ribonuclease takes roughly 4672 seconds (or 1 hr, 18 min) to cross a distance of 1 mm by diffusion alone. These examples illustrate why organisms frequently rely on other active mechanisms to transport molecules.

The diffusion equation (10.35) can be generalized to higher dimensions. In that case, (10.33) becomes

$$\frac{\partial c}{\partial t} = -\nabla J \tag{10.39}$$

and (10.34) becomes

$$J = -D\nabla c \tag{10.40}$$

Combining (10.39) and (10.40), we find that

$$\frac{\partial c}{\partial t} = D\nabla \cdot (\nabla c)$$

where $\nabla \cdot (\nabla c)$ is to be interpreted as a dot product. That is, if $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}$, then

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x_1^2} + \frac{\partial^2 c}{\partial x_2^2} + \frac{\partial^2 c}{\partial x_3^2}\right)$$

As a shorthand notation, we define

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

where Δ is called the **Laplace operator**. We then write

$$\frac{\partial c}{\partial t} = D\,\Delta c$$

More generally, if $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, then

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

(Δc is read "the Laplacian of c.")

Section 10.6 Problems

10.6.1

In Problems 1–10, the functions are defined for all $(x, y) \in \mathbf{R}^2$. Find all candidates for local extrema, and use the Hessian matrix to determine the type (maximum, minimum, or saddle point).

1. $f(x, y) = x^2 + y^2 - 2x$
2. $f(x, y) = -2x^2 - y^2 + 3y$
3. $f(x, y) = x^2y - 4x^2 - 4y$
4. $f(x, y) = xy - 2y^2$
5. $f(x, y) = -2x^2 + y^2 - 6y$
6. $f(x, y) = x(1 - x + y)$
7. $f(x, y) = e^{-x^2-y^2}$
8. $f(x, y) = yxe^{-y}$
9. $f(x, y) = x\cos y$
10. $f(x, y) = y\sin x$

11. In this problem, we will illustrate that if one of the eigenvalues of the Hessian matrix at a point where the gradient vanishes is equal to 0, then we cannot make any statements about whether the point is a local extremum just on the basis of the Hessian matrix. Consider the following functions:

$$f_1(x, y) = x^2$$
$$f_2(x, y) = x^2 + y^3$$
$$f_3(x, y) = x^2 + y^4$$

Figures 10.68 through 10.70 show graphs of the three functions.

(a) Show that, for $i = 1, 2$, and 3,

$$\nabla f_i(0, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) Show that, for $i = 1, 2$, and 3,

$$\text{Hess}\, f_i(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

and determine the eigenvalues of Hess $f_i(0, 0)$.

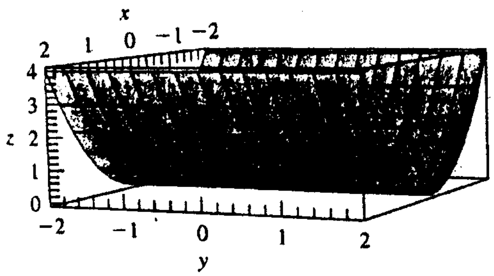

Figure 10.68 $f_1(x, y)$ in Problem 11.

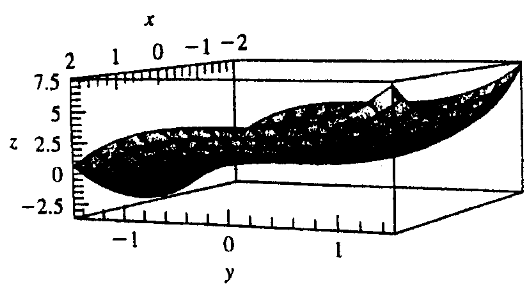

Figure 10.69 $f_2(x, y)$ in Problem 11.

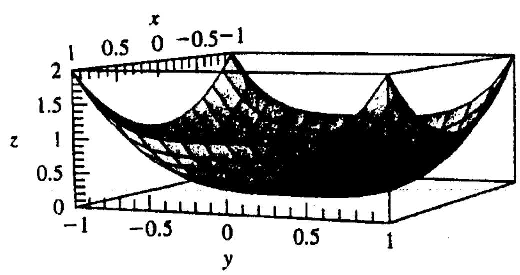

Figure 10.70 $f_3(x, y)$ in Problem 11.

(c) Since one of the eigenvalues of Hess $f_i(0, 0)$ is equal to 0, we cannot use the criterion stated in the text to determine the behavior of the three functions at (0, 0). Use Figures 10.68 through 10.70 to describe what happens at (0, 0) for each function.

12. Consider the function

$$f(x, y) = ax^2 + by^2$$

(a) Show that

$$\nabla f(0, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) Find values for a and b such that **(i)** (0, 0) is a local minimum, **(ii)** (0, 0) is a local maximum, and **(iii)** (0, 0) is a saddle point.

In Problems 13–16, the functions are defined on the rectangular domain

$$D = \{(x, y) : -1 \le x \le 1, -1 \le y \le 1\}$$

Find the absolute maxima and minima of f on D.

13. $f(x, y) = 2x - y$

14. $f(x, y) = 3 - x + 2y$

15. $f(x, y) = x^2 - y^2$

16. $f(x, y) = x^2 + y^2$

17. Find the absolute maxima and minima of

$$f(x, y) = x^2 + y^2 - x + 2y$$

on the set

$$D = \{(x, y) = 0 \le x \le 1, -2 \le y \le 0\}$$

18. Find the absolute maxima and minima of

$$f(x, y) = x^2 - y^2 + 4x + y$$

on the set

$$D = \{(x, y) = -4 \le x \le 0, 0 \le y \le 1\}$$

19. Maximize the function

$$f(x, y) = 2xy - x^2y - xy^2$$

on the triangle bounded by the line $x + y = 2$, the x-axis, and the y-axis.

20. Maximize the function

$$f(x, y) = xy(15 - 5y - 3x)$$

on the triangle bounded by the line $5y + 3x = 15$, the x-axis, and the y-axis.

21. Find the absolute maxima and minima of

$$f(x, y) = x^2 + y^2 + 4x - 1$$

on the disk

$$D = \{(x, y) : x^2 + y^2 \le 9\}$$

22. Find the absolute maxima and minima of

$$f(x, y) = x^2 + y^2 - 6y + 3$$

on the disk

$$D = \{(x, y) : x^2 + y^2 \le 16\}$$

23. Find the absolute maxima and minima of

$$f(x, y) = x^2 + y^2 + x - y$$

on the disk

$$D = \{(x, y) : x^2 + y^2 \le 1\}$$

24. Find the absolute maxima and minima of

$$f(x, y) = x^2 + y^2 + x + 2y$$

on the disk

$$D = \{(x, y) : x^2 + y^2 \le 4\}$$

25. Can a continuous function of two variables have two maxima and no minima? Describe in words what the properties of such a function would be, and contrast this behavior with a function of one variable.

26. Suppose $f(x, y)$ has a horizontal tangent plane at (0, 0). Can you conclude that f has a local extremum at (0, 0)?

27. Suppose crop yield Y depends on nitrogen (N) and phosphorus (P) concentrations as

$$Y(N, P) = NPe^{-(N+P)}$$

Find the value of (N, P) that maximizes crop yield.

28. Choose three numbers x, y, and z so that their sum is equal to 60 and their product is maximal.

29. Find the maximum volume of a rectangular closed (top, bottom, and four sides) box with surface area 48 m^2.

30. Find the maximum volume of a rectangular open (bottom and four sides, no top) box with surface area 75 m^2.

31. Find the minimum surface area of a rectangular closed (top, bottom, and four sides) box with volume 216 m^3.

32. Find the minimum surface area of a rectangular open (bottom and four sides, no top) box with volume 256 m^3.

33. The distance between the origin (0, 0, 0) and the point (x, y, z) is

$$\sqrt{x^2 + y^2 + z^2}$$

Find the minimum distance between the origin and the plane $x + y + z = 1$. (*Hint*: Minimize the squared distance between the origin and the plane.)

34. Given the symmetric matrix

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

where a, b, and c are real numbers, show that the eigenvalues of A are real. (*Hint*: Compute the eigenvalues.)

35. Understanding species richness and diversity is a major concern of ecological studies. A frequently used measure of diversity is the Shannon and Weaver index

$$H = -\sum_{i=1}^{n} p_i \ln p_i$$

where p_i is equal to the proportion of species i, $i = 1, 2, \ldots, n$, and n is the total number of species in the study area. Assume that a community consists of three species with relative proportions p_1, p_2, and p_3.

(a) Use the fact that $p_1 + p_2 + p_3 = 1$ to show that H is of the form

$$H(p_1, p_2) = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2)\ln(1 - p_1 - p_2)$$

and that the domain of $H(p_1, p_2)$ is the triangular set in the p_1–p_2 plane bounded by the lines $p_1 = 0$, $p_2 = 0$, and $p_1 + p_2 = 1$.

(b) Show that H attains its absolute maximum when $p_1 = p_2 = p_3 = 1/3$.

■ 10.6.2

In Problems 36–45, use Lagrange multipliers to find the maxima and minima of the functions under the given constraints.

36. $f(x, y) = 2x - y; x^2 + y^2 = 5$

37. $f(x, y) = 3x^2 + y; x^2 + y^2 = 1$

38. $f(x, y) = xy; x^2 + y^2 = 4$

39. $f(x, y) = xy; 2x - 4y = 1$

40. $f(x, y) = x^2 - y^2; 2x + y = 1$

41. $f(x, y) = x^2 + y^2; 3x - 2y = 4$

42. $f(x, y) = xy^2; x^2 - y = 0$

43. $f(x, y) = x^2y; x^2 + 3y = 1$

44. $f(x, y) = x^2y^2; 2x - 3y = 4$

45. $f(x, y) = x^2y^2; x^2 - y^2 = 1$

In Problems 46–55, use Lagrange multipliers to find the answers to the indicated problems in Section 5.4.

46. Problem 1 **47.** Problem 2

48. Problem 3 **49.** Problem 4

50. Problem 5 **51.** Problem 6

52. Problem 7 **53.** Problem 9

54. Problem 12 **55.** Problem 18

56. Let

$$f(x, y) = x + y \quad (x, y) \in \mathbf{R}^2$$

with constraint function $xy = 1$.

(a) Use Lagrange multipliers to find all local extrema.

(b) Are there global extrema?

57. Let

$$f(x, y) = x + y$$

with constraint function

$$\frac{1}{x} + \frac{1}{y} = 1, \; x \neq 0, y \neq 0$$

(a) Use Lagrange multipliers to find all local extrema.

(b) Are there global extrema?

58. Let

$$f(x, y) = xy, \quad (x, y) \in \mathbf{R}^2$$

with constraint function $y - x^2 = 0$.

(a) Use Lagrange multipliers to find candidates for local extrema.

(b) Use the constraint $y - x^2 = 0$ to reduce $f(x, y)$ to a single-variable function, and then use this function to show that $f(x, y)$ has no local extrema on the constraint curve.

59. Explain why $f(x, y)$ has a local extremum at the point P in Figure 10.63 under the constraint $g(x, y) = 0$ if $c_1 > c_2 > c_3 > c_4$.

60. Explain why $f(x, y)$ has a local extremum at the point P in Figure 10.63 under the constraint $g(x, y) = 0$ if $c_1 < c_2$ and $c_2 > c_3 > c_4$.

61. In the introductory example, we discussed how egg size depends on maternal age. Assume now that the total amount of resources available is 10 (in appropriate units), the number of eggs per clutch is 3, the number of clutches is 2, and the egg size in clutch number i is denoted by x_i.

(a) Find the constraint function.

(b) Suppose the fitness function is given by

$$f(x_1, x_2) = \frac{3}{2}\rho(x_1) + \frac{3}{4}\rho(x_2)$$

where $\rho(x) = \frac{2x}{5+x}$. Find the optimal egg sizes for clutch 1 and clutch 2 under the constraint in (a).

62. In the introductory example in this subsection, we discussed how egg size depends on maternal age. Assume now that the fitness function is given by

$$f(x_1, x_2) = \frac{5}{3}\rho(x_1) + \frac{5}{6}\rho(x_2)$$

with

$$\rho(x) = \frac{3x}{4 + x}$$

The constraint function is given by

$$5x_1 + 5x_2 = 7$$

(a) Compare the given functions with the corresponding ones in the text, and identify the parameters n, p_1, p_2, and R from the text.

(b) Solve the constraint function for x_2 and substitute your expression for x_2 into the function f. This then yields a function of one variable. Find the domain of this single-variable function and use single-variable calculus to determine optimal egg sizes for clutch 1 and clutch 2.

■ 10.6.3

63. Show that

$$c(x, t) = \frac{1}{\sqrt{8\pi t}} \exp\left[-\frac{x^2}{8t}\right]$$

solves

$$\frac{\partial c(x, t)}{\partial t} = 2\frac{\partial^2 c(x, t)}{\partial x^2}$$

64. Show that

$$c(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right]$$

solves

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 c(x,t)}{\partial x^2}$$

65. A solution of

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}$$

is the function

$$c(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{x^2}{4Dt}\right]$$

for $x \in \mathbf{R}$ and $t > 0$.

(a) Show that, as a function of x for fixed values of $t > 0$, $c(x,t)$ is **(i)** positive for all $x \in \mathbf{R}$, **(ii)** is increasing for $x < 0$ and decreasing for $x > 0$, **(iii)** has a local maximum at $x = 0$, and **(iv)** has inflection points at $x = \pm\sqrt{2Dt}$.

(b) Graph $c(x,t)$ as a function of x when $D = 1$ for $t = 0.01$, $t = 0.1$, and $t = 1$.

66. A solution of

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}$$

is the function

$$c(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{x^2}{4Dt}\right]$$

for $x \in \mathbf{R}$ and $t > 0$.

(a) Show that a local maximum of $c(x,t)$ occurs at $x = 0$ for fixed t.

(b) Show that $c(0,t)$, $t > 0$, is a decreasing function of t.

(c) Find

$$\lim_{t \to 0^+} c(x,t)$$

when $x = 0$ and when $x \neq 0$.

(d) Use the fact that

$$\int_{-\infty}^{\infty} e^{-u^2/2}\,du = \sqrt{2\pi}$$

to show that, for $t > 0$,

$$\int_{-\infty}^{\infty} c(x,t)\,dx = 1$$

(e) The function $c(x,t)$ can be interpreted as the concentration of a substance diffusing in space. Explain the meaning of

$$\int_{-\infty}^{\infty} c(x,t)\,dx = 1$$

and use your results in (c) and (d) to explain why this means that initially (i.e., at $t = 0$) the entire amount of the substance was released at the origin.

Mathematically, we can specify such an initial condition (in which the substance is concentrated at the origin at time 0) by the δ-function $\delta(x)$, with the property that

$$\delta(x) = 0, \quad \text{for } x \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

67. The two-dimensional diffusion equation

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = D\left(\frac{\partial^2 n(\mathbf{r},t)}{\partial x^2} + \frac{\partial^2 n(\mathbf{r},t)}{\partial y^2}\right) \tag{10.41}$$

where $n(\mathbf{r},t)$, $\mathbf{r} = (x,y)$, denotes the population density at the point $\mathbf{r} = (x,y)$ in the plane at time t, can be used to describe the spread of organisms. Assume that a large number of insects are released at time 0 at the point (0, 0). Furthermore, assume that, at later times, the density of these insects can be described by the diffusion equation (10.41). Show that

$$n(x,y,t) = \frac{n_0}{4\pi Dt} \exp\left[-\frac{x^2+y^2}{4Dt}\right]$$

satisfies (10.41).

■ 10.7 Systems of Difference Equations (Optional)

■ 10.7.1 A Biological Example

About 14% of all insect species (and thus about 10% of all species of multicellular animals) are estimated to belong to a group of insects called *parasitoids*. These are insects (mostly in the order Hymenoptera) that lay their eggs on, in, or near the (in most cases, immature) body of another arthropod, which serves as a host for the developing parasitoids. The eggs develop into free-living adults while consuming the host.

Parasitoids play an important role in biological control. A successful example is *Trichogramma* wasps, which parasitize insect eggs. These wasps are reared in factories for subsequent release to the field. Every year, millions of hectares of agricultural land are treated with released *Trichogramma* wasps, for instance, to protect sugar cane from the sugarcane borer, *Chilo* spp., in China, or to protect cornfields from the European corn borer, *Ostrinia nubilalis* (Hübner), in western Europe. Another successful example of biological control of an insect pest is the

The solutions of this equation are complex conjugate if the discriminant

$$\left(1 + \frac{ac}{b}N^*\right)^2 - 4acN^* < 0$$

With $N^* = \frac{b}{b-1}\frac{1}{ac}\ln b$, the discriminant is

$$f(b) = \left(1 + \frac{\ln b}{b-1}\right)^2 - \frac{4b}{b-1}\ln b$$

This function depends only on b. Graphing $f(b)$ (see Figure 10.74) shows that $f(b) < 0$ for $b > 1$, thus confirming that the two eigenvalues of J are complex conjugate if $b > 1$.

When we discussed linear systems of difference equations, we derived the identity

$$|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1\lambda_2 = \det J$$

The determinant of J is given by

$$\det J = acN^*\frac{1}{b} + acN^*\left(1 - \frac{1}{b}\right) = acN^* = \frac{b\ln b}{b-1}$$

Graphing $g(b) = \frac{b\ln b}{b-1}$ as a function of b (see Figure 10.75), we see that $g(b) > 1$ for $b > 1$, from which we conclude that, for $b > 1$,

$$|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1\lambda_2 > 1$$

implying that the nontrivial equilibrium is unstable. ■

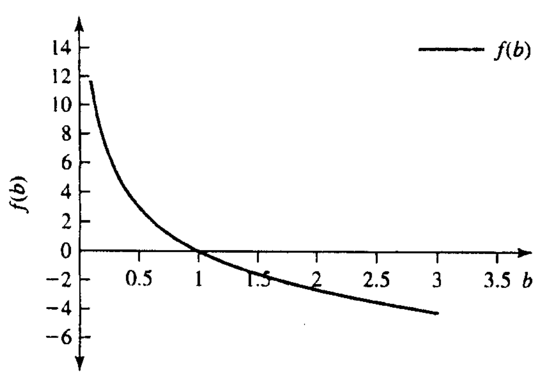

Figure 10.74 The graph of $f(b)$ confirms that $f(b) < 0$ for $b > 1$.

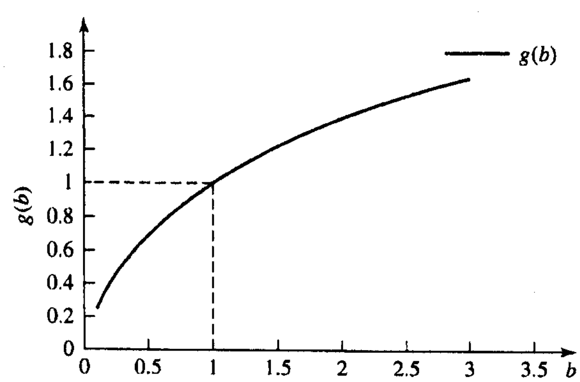

Figure 10.75 The graph of $g(b)$ confirms that $g(b) > 1$ for $b > 1$.

Section 10.7 Problems

10.7.1

'roblems 1–6 refer to the Nicholson–Bailey host-parasitoid model. 'roblems 1, 2, 5, and 6 are best done with the help of a spreadsheet, ut can also be done with a calculator. Nicholson and Bailey itroduced the discrete-generation host-parasitoid model of the ›rm

$$N_{t+1} = bN_te^{-aP_t}$$
$$P_{t+1} = cN_t[1 - e^{-aP_t}]$$

›r $t = 0, 1, 2, \ldots$

1. Evaluate the Nicholson–Bailey model for the first 10 generations when $a = 0.02$, $c = 3$, and $b = 1.5$. For the initial host density, choose $N_0 = 5$, and for the initial parasitoid density, choose $P_0 = 0$.

2. Evaluate the Nicholson–Bailey model for the first 10 generations when $a = 0.02$, $c = 3$, and $b = 0.5$. For the initial host density, choose $N_0 = 15$, and for the initial parasitoid density, choose $P_0 = 0$.

3. Show that when the initial parasitoid density is $P_0 = 0$, the Nicholson–Bailey model reduces to

$$N_{t+1} = bN_t$$

With N_0 denoting the initial host density, find an expression for N_t in terms of N_0 and the parameter b.

4. When the initial parasitoid density is $P_0 = 0$, the Nicholson–Bailey model reduces to

$$N_{t+1} = bN_t$$

as shown in the previous problem. For which values of b is the host density increasing if $N_0 > 0$? For which values of b is it decreasing? (Assume that $b > 0$.)

5. Evaluate the Nicholson-Bailey model for the first 15 generations when $a = 0.02$, $c = 3$, and $b = 1.5$. For the initial host density, choose $N_0 = 5$, and for the initial parasitoid density, choose $P_0 = 5$.

6. Evaluate the Nicholson-Bailey model for the first 25 generations when $a = 0.02$, $c = 3$, and $b = 1.5$. For the initial host density, choose $N_0 = 15$, and for the initial parasitoid density, choose $P_0 = 8$.

Problems 7–12 refer to the negative binomial host–parasitoid model. Problems 7, 8, 11, and 12 are best done with the help of a spreadsheet, but can also be done with a calculator. The negative binomial model is a discrete-generation host–parasitoid model of the form

$$N_{t+1} = bN_t\left(1 + \frac{aP_t}{k}\right)^{-k}$$

$$P_{t+1} = cN_t\left[1 - \left(1 + \frac{aP_t}{k}\right)^{-k}\right]$$

for $t = 0, 1, 2, \ldots$.

7. Evaluate the negative binomial model for the first 10 generations when $a = 0.02$, $c = 3$, $k = 0.75$, and $b = 1.5$. For the initial host density, choose $N_0 = 5$, and for the initial parasitoid density, choose $P_0 = 0$.

8. Evaluate the negative binomial model for the first 10 generations when $a = 0.02$, $c = 3$, $k = 0.75$, and $b = 0.5$. For the initial host density, choose $N_0 = 15$, and for the initial parasitoid density, choose $P_0 = 0$.

9. Show that when the initial parasitoid density is $P_0 = 0$, the negative binomial model reduces to

$$N_{t+1} = bN_t$$

With N_0 denoting the initial host density, find an expression for N_t in terms of N_0 and the parameter b.

10. When the initial parasitoid density is $P_0 = 0$, the negative binomial model reduces to

$$N_{t+1} = bN_t$$

as shown in the previous problem. For which values of b is the host density increasing if $N_0 > 0$? For which values of b is it decreasing? (Assume that $b > 0$.)

11. Evaluate the negative binomial model for the first 25 generations when $a = 0.02$, $c = 3$, $k = 0.75$, and $b = 1.5$. For the initial host density, choose $N_0 = 100$, and for the initial parasitoid density, choose $P_0 = 50$.

12. Evaluate the negative binomial model for the first 25 generations when $a = 0.02$, $c = 3$, $k = 0.75$, and $b = 0.5$. For the initial host density, choose $N_0 = 100$, and for the initial parasitoid density, choose $P_0 = 50$.

13. In the Nicholson–Bailey model, the fraction of hosts escaping parasitism is given by

$$f(P) = e^{-aP}$$

(a) Graph $f(P)$ as a function of P for $a = 0.1$ and $a = 0.01$.
(b) For a given value of P, how are the chances of escaping parasitism affected by increasing a?

14. In the negative binomial model, the fraction of hosts escaping parasitism is given by

$$f(P) = \left(1 + \frac{aP}{k}\right)^{-k}$$

(a) Graph $f(P)$ as a function of P for $a = 0.1$ and $a = 0.01$ when $k = 0.75$.
(b) For $k = 0.75$ and a given value of P, how are the chances of escaping parasitism affected by increasing a?

15. In the negative binomial model, the fraction of hosts escaping parasitism is given by

$$f(P) = \left(1 + \frac{aP}{k}\right)^{-k}$$

(a) Graph $f(P)$ as a function of P for $k = 0.75$ and $k = 0.5$ when $a = 0.02$.
(b) For $a = 0.02$ and a given value of P, how are the chances of escaping parasitism affected by increasing k?

16. The negative binomial model can be reduced to the Nicholson–Bailey model by letting the parameter k in the negative binomial model go to infinity. Show that

$$\lim_{k\to\infty}\left(1 + \frac{aP}{k}\right)^{-k} = e^{-aP}$$

(*Hint*: Use l'Hospital's rule.)

■ 10.7.2

17. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix} = \begin{bmatrix}-0.7 & 0\\-0.3 & 0.2\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and determine its stability.

18. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix} = \begin{bmatrix}0.4 & 0.2\\0 & -0.9\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and determine its stability.

19. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix} = \begin{bmatrix}-1.4 & 0\\-0.5 & 0.1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and determine its stability.

20. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix} = \begin{bmatrix}0.1 & 0.4\\0.1 & -0.2\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and determine its stability.

21. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and determine its stability.

22. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix}=\begin{bmatrix}1.5 & 0.2\\0.08 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and determine its stability.

23. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix}=\begin{bmatrix}-0.2 & -0.4\\0.6 & 0.1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

is stable.

24. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix}=\begin{bmatrix}0.2 & 0.3\\-0.5 & -0.4\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

is stable.

25. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix}=\begin{bmatrix}4.2 & -3.4\\2.4 & -1.1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

is unstable.

26. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix}=\begin{bmatrix}2 & -4\\5 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

is stable.

■ 10.7.3

27. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$x_1(t+1)=\frac{x_2(t)}{4(1+x_1^2(t))}$$
$$x_2(t+1)=\frac{2x_1(t)}{1+x_2^2(t)}$$

is locally stable.

28. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$x_1(t+1)=\frac{3x_2(t)}{1+x_1^2(t)}$$
$$x_2(t+1)=\frac{2x_1(t)}{1+x_2^2(t)}$$

s unstable.

29. Show that the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$x_1(t+1)=x_2(t)$$
$$x_2(t+1)=\frac{2x_2(t)-x_1(t)}{2+x_1(t)}$$

s locally stable.

30. Show that, for any $a>1$, the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$x_1(t+1)=x_2(t)$$
$$x_2(t+1)=\frac{ax_2(t)-(a-1)x_1(t)}{a+x_1(t)}$$

s locally stable.

31. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ is an equilibrium point of

$$x_1(t+1)=ax_2(t)$$
$$x_2(t+1)=2x_1(t)-\cos(x_2(t))+1$$

Assume that $a>0$. For which values of a is $\begin{bmatrix}0\\0\end{bmatrix}$ locally stable?

32. Show that $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}-\pi\\\pi\end{bmatrix}$ are equilibria of

$$x_1(t+1)=-x_2(t)$$
$$x_2(t+1)=\sin(x_2(t))-x_1(t)$$

and analyze their stability.

33. Find all nonnegative equilibria of

$$x_1(t+1)=x_2(t)$$
$$x_2(t+1)=\frac{1}{2}x_1(t)+\frac{2}{3}x_2(t)-x_2^2(t)$$

and analyze their stability.

34. Find all nonnegative equilibria of

$$x_1(t+1)=x_2(t)$$
$$x_2(t+1)=\frac{1}{2}x_1(t)+\frac{1}{3}x_2(t)-x_2^2(t)$$

and analyze their stability.

35. For which values of a is the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$x_1(t+1)=\frac{ax_2(t)}{1+x_1^2(t)}$$
$$x_2(t+1)=\frac{x_1(t)}{1+x_2^2(t)}$$

locally stable?

36. For which values of a is the equilibrium $\begin{bmatrix}0\\0\end{bmatrix}$ of

$$x_1(t+1)=x_2(t)$$
$$x_2(t+1)=\frac{1}{2}x_1(t)+ax_2(t)-x_2^2(t)$$

locally stable?

37. Denote by $x_1(t)$ the number of juveniles, and by $x_2(t)$ the number of adults, at time t. Assume that $x_1(t)$ and $x_2(t)$ evolve according to

$$x_1(t+1)=x_2(t)$$
$$x_2(t+1)=\frac{1}{2}x_1(t)+rx_2(t)-x_2^2(t)$$

(a) Show that if $r>1/2$, there exists an equilibrium $\begin{bmatrix}x_1^*\\x_2^*\end{bmatrix}$ with $x_1^*>0$ and $x_2^*>0$. Find x_1^* and x_2^*.

(b) Determine the stability of the equilibrium found in (a) when $r>1/2$.

38. Find all biologically relevant equilibria of the Nicholson–Bailey model

$$N_{t+1} = 2N_t e^{-0.2P_t}$$
$$P_{t+1} = N_t\left[1 - e^{-0.2P_t}\right]$$

and analyze their stability.

39. Find all biologically relevant equilibria of the Nicholson–Bailey model

$$N_{t+1} = 4N_t e^{-0.1P_t}$$
$$P_{t+1} = N_t\left[1 - e^{-0.1P_t}\right]$$

and analyze their stability.

40. Find all biologically relevant equilibria of the negative binomial host–parasitoid model

$$N_{t+1} = 4N_t\left(1 + \frac{0.01P_t}{2}\right)^{-2}$$
$$P_{t+1} = N_t\left[1 - \left(1 + \frac{0.01P_t}{2}\right)^{-2}\right]$$

and analyze their stability.

41. Find all biologically relevant equilibria of the negative binomial host–parasitoid model

$$N_{t+1} = 4N_t\left(1 + \frac{0.01P_t}{0.5}\right)^{-0.5}$$
$$P_{t+1} = N_t\left[1 - \left(1 + \frac{0.01P_t}{0.5}\right)^{-0.5}\right]$$

and analyze their stability.

Chapter 10 Key Terms

Discuss the following definitions and concepts:

1. Real-valued function
2. Function of two variables
3. Surface
4. Level curve
5. Limit
6. Limit laws
7. Continuity
8. Partial derivative
9. Geometric interpretation of a partial derivative
10. Mixed-derivative theorem
11. Tangent plane
12. Differentiability
13. Differentiability and continuity
14. Sufficient condition for differentiability
15. Standard linear approximation, tangent plane approximation
16. Vector-valued function
17. Jacobi matrix, derivative matrix
18. Chain rule
19. Implicit differentiation
20. Directional derivative
21. Gradient
22. Local extrema
23. Sufficient condition for finding local extrema
24. Hessian matrix
25. Global extrema
26. The extreme-value theorem
27. Diffusion
28. Systems of difference equations
29. Point equilibria and their stability
30. Nicholson–Bailey equation

Chapter 10 Review Problems

1. Germination Suppose that you conduct an experiment to measure the germination success of seeds of a certain plant as a function of temperature and humidity. You find that seeds don't germinate at all when the humidity is too low, regardless of temperature; germination success is highest for intermediate values of temperature; and seeds tend to germinate better when you increase humidity levels. Use the preceding information to sketch a graph of germination success as a function of temperature for different levels of humidity. Also, sketch the graph of germination success as a function of humidity for different temperature values.

2. Plant Physiology Gaastra (1959) measured the effects of atmospheric CO_2 enrichment on CO_2 fixation in sugar beet leaves at various light levels. He found that increasing CO_2 at fixed light levels increases the fixation rate and that increasing light levels at fixed atmospheric CO_2 concentration also increased fixation. If $F(A, I)$ denotes the fixation rate as a function of atmospheric CO_2 concentration (A) and light intensity (I), determine the signs of $\partial F/\partial A$ and $\partial F/\partial I$.

3. Plant Ecology In Burke and Grime (1996), a long-term field experiment in a limestone grassland was described.

(a) One of the experiments related total area covered by *indigenous* species to fertility and disturbance gradients. The experiment was designed so that the two variables (fertility and disturbance) could be altered independently. Burke and Grime found that the area covered by indigenous species generally increased with the amount of fertilizer added and decreased with the intensity of a disturbance. If $A_i(F, D)$ denotes the area covered by indigenous species as a function of the amount of fertilizer added (F) and the intensity of disturbance (D), determine the signs of $\partial A_i/\partial F$ and $\partial A_i/\partial D$ for Burke and Grime's experiment.

(b) In another experiment, Burke and Grime related the total area covered by *introduced* species to fertility and disturbance gradients. Let $A_e(F, D)$ denote the area covered by introduced species as a function of the amount of fertilizer added (F) and the intensity of disturbance (D). Burke and Grime found that

$$\frac{\partial A_e}{\partial F} > 0$$

and

$$\frac{\partial A_e}{\partial D} > 0$$

Explain in words what this means.

(c) Compare the responses to fertilization and disturbance with the area covered in the two experiments.

4. Plant Physiology Vitousek and Farrington (1997) investigated nutrient limitations in soils of different ages. In the abstract of their paper, they say,

> Walker and Syers (1976) proposed a conceptual model that describes the pattern and regulation of soil nutrient pools and availability during long-term soil and ecosystem development. Their model implies that plant production generally should be limited by N [nitrogen] on young soils and by P [phosphorus] on old soils; N and P supply should be more or less equilibrate on intermediate aged soils.

Vitousek and Farrington tested this hypothesis by conducting fertilizer experiments along a gradient of soil age, measuring the average increment in diameter (in mm/yr) of *Metrosideros polymorpha* trees.

Denote by $D(\mathrm{N}, \mathrm{P}, t)$ the diameter increment (in mm/yr) as a function of the amount of nitrogen (N) added, the amount of phosphorus (P) added, and the age (t) of the soil. Vitousek and Farrington's experiments showed that

$$\frac{\partial D}{\partial t}(\mathrm{N}, 0, t) < 0$$

and

$$\frac{\partial D}{\partial t}(0, \mathrm{P}, t) > 0$$

for their choices of $\mathrm{N} > 0$ and $\mathrm{P} > 0$. Explain why their results support the Walker and Syers hypothesis.

5. Find the Jacobi matrix

$$\mathbf{f}(x, y) = \begin{bmatrix} x^2 - y \\ x^3 - y^2 \end{bmatrix}$$

6. Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} 2xy^2 \\ \frac{x}{y} \end{bmatrix}$$

at $(1, 1)$.

7. Mark–Recapture Experiment We can compute the average radius of spreading individuals at time t, denoted by r_{avg}. We find that

$$r_{\text{avg}} = \sqrt{\pi D t} \tag{10.55}$$

(a) Graph r_{avg} as a function of D for $t = 0.1$, $t = 1$, and $t = 5$. Describe in words how an increase in D affects the average radius of spread.

(b) Show that

$$D = \frac{(r_{\text{avg}})^2}{\pi t} \tag{10.56}$$

(c) Equation (10.56) can be used to measure D, the diffusion constant, from field data of mark–recapture experiments, taken from Kareiva (1983), as follows: Marked organisms are released from the release site and then recaptured after a certain amount of time t from the time of release. The distance of the recaptured organisms from the release site is measured.

If N denotes the total number of recaptured organisms, d_i denotes the distance of the ith recaptured organism from the release site, and t is the time between release and recapture, use (10.56) to explain why

$$D = \frac{1}{\pi t}\left(\frac{1}{N}\sum_{i=1}^{N} d_i\right)^2$$

can be used to measure D from field data. (Note that the time between release and recapture is the same for each individual in this study.)