Math 17C
Kouba

Solving Linear Systems of Differential Equations Having Complex Eigenvalues

Ex: Solve $X' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X$; Find eigenvalues:

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - (-1)(1)$$

$$= \lambda^2 - 2\lambda + 1 + 1 = \lambda^2 - 2\lambda + 2 = 0 \longrightarrow$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i \quad ;$$

Solve $(A-\lambda I)X = 0$ for X :

For $\lambda = 1+i$: $\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -i & 0 \end{array}\right] \rightarrow x_1 - i x_2 = 0$

so let $x_2 = t$ any # $\longrightarrow$ $x_1 = i x_2 = it \longrightarrow$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$ so eigenvector is $\begin{bmatrix} i \\ 1 \end{bmatrix}$;

then
$$X = c\begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1+i)t} = c\begin{bmatrix} i \\ 1 \end{bmatrix} e^{t} e^{it}$$

$$= c\begin{bmatrix} i \\ 1 \end{bmatrix} e^{t} (\cos t + i \sin t)$$

$$= c\begin{bmatrix} i\cos t - \sin t \\ \cos t + i \sin t \end{bmatrix} e^{t}$$

$$= c\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^{t} + ci\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{t}$$

$$= c_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^{t} + c_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{t} .$$