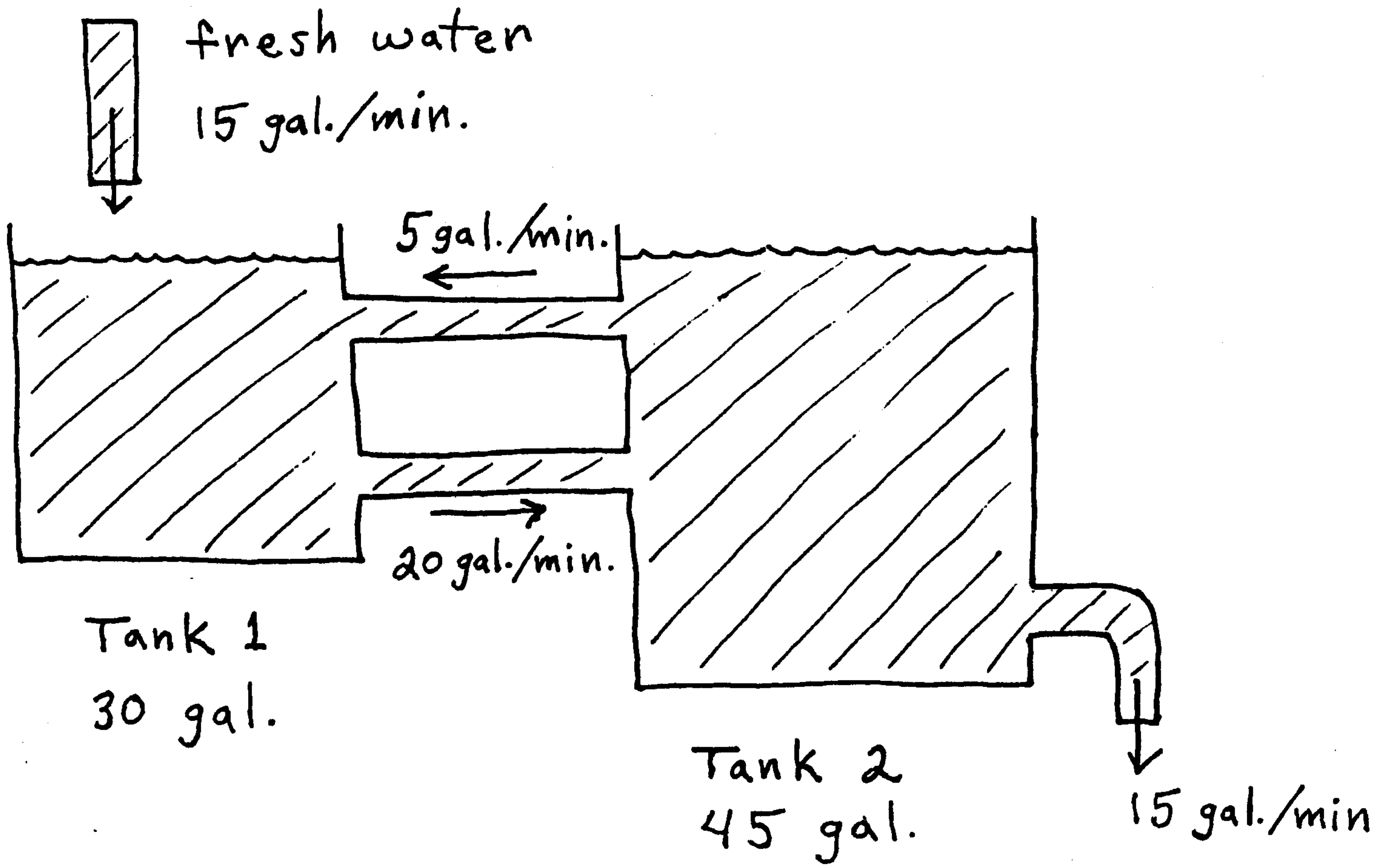
Math 17C
Kouba
Discussion Sheet 6

1.) Consider the two tanks containing salt water solutions and connected as shown in the diagram. Tank 1 holds 30 gallons of salt water solution. Tank 2 holds 45 gallons of salt water solution. Let x_1 and x_2 represent the pounds of salt in Tank 1 and Tank 2, resp., at time t. Initially, Tank 1 contains 10 pounds of salt and Tank 2 contains 25 pounds of salt. The mixture in each tank is kept uniform by stirring, and the mixtures are pumped from each tank to the other at the rates indicated in the figure. In addition, fresh water is pumped into Tank 1 at the rate of 15 gal./min.; the mixture leaves Tank 2 at 15 gal./min. Set up and solve a system of differential equations with initial conditions, which represents the amount of salt in each tank. How much salt is in each tank $t = 10$ minutes ?

Let x_1 : lbs. of salt in Tank 1 at time t
x_2 : lbs. of salt in Tank 2 at time t
t : minutes

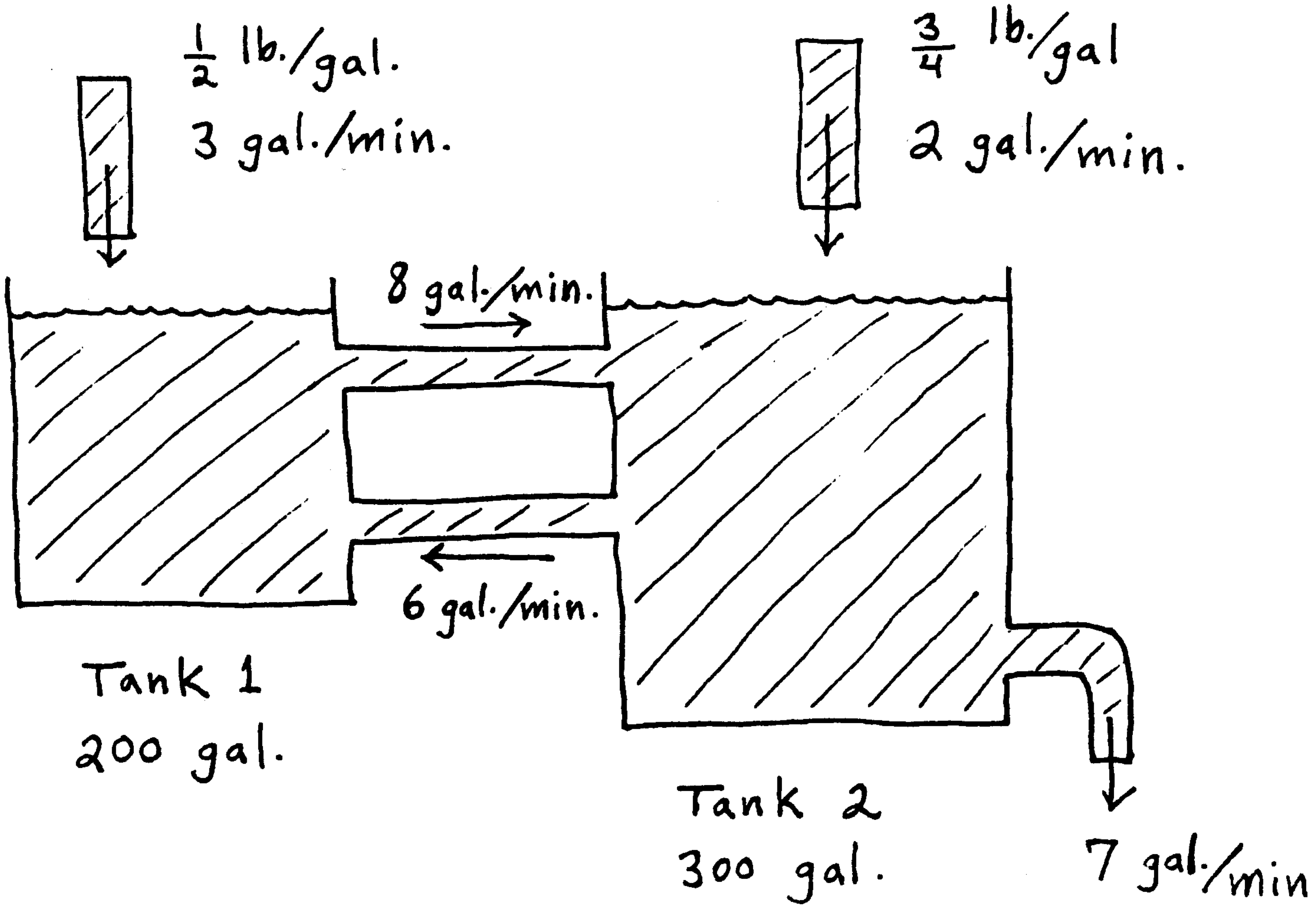


2.) Consider the two tanks containing Snapple tea and honey mixtures and connected as shown in the diagram. Tank 1 holds 200 gallons of mixture. Tank 2 holds 300 gallons of mixture. Let x_1 and x_2 represent the pounds of honey in Tank 1 and Tank 2, resp., at time t. Initially, Tank 1 contains 60 pounds of honey and Tank 2 contains 100 pounds of honey. The mixture in each tank is kept uniform by stirring, and the mixtures are pumped from each tank to the other at the rates indicated in the figure. In addition, a mixture containing 1/2 pound of honey per gallon is pumped into Tank 1 at 3 gal./min.; a mixture containing 3/4 pound of honey per gallon is pumped into Tank 2 at 2 gal./min.; the mixture leaves Tank 2 at 7 gal./min. SET UP, BUT DO NOT SOLVE, a sytem of differential equations with initial conditions, which represents the amount of honey in each tank. PAY CLOSE ATTENTION TO FLOW RATES IN AND OUT OF EACH TANK !!

Let x_1: lbs. of honey in Tank 1 at time t

x_2: lbs. of honey in Tank 2 at time t

t: minutes



For problems 3 and 4 let x_1 and x_2 be the amount of drug in the person's body tissue and urinary tract, resp., at time t hours. Set up a system of D.E.'s using our Drug Dissipation Model from class and then solve for x_1 and x_2 and answer the given questions.

3.) A single dose of quinine sulfate (an antimalarial drug) is administered to a patient. A laboratory measurement shows that the urinary tract has accumulated 171 mg. of the drug after 5 hours. Let x_1 and x_2 be the mg. of quinine sulfate in the person's body tissue and urinary tract, resp., at time t hours and assume that $\frac{dx_1}{dt} = -.15x_1$. Determine the initial dosage in mg. of quinine sulphate.

4.) Assume that a 21-year old college student ingests four consecutive shots of liquor containing a total of 2 ounces of ethanol (grain alcohol). After 1 hour there is 1 ounce of ethanol in the person's body tissue. Assume that the rate at which ethanol leaves the body tissue is proportional to the square of the the amount of ethanol in the body tissue. How much ethanol has passed into the urinary tract after $t = 5$ hours ?

5.) Find the Jacobi Matrix for each function.

a.) $f(x,y) = \begin{pmatrix} 3x+4y \\ 2x-5y \end{pmatrix}$ b.) $f(x,y) = \begin{pmatrix} x+e^y \\ \ln(x-y) \end{pmatrix}$

c.) $f(x,y) = \begin{pmatrix} x/y \\ x^3 \sin y \end{pmatrix}$ d.) $f(x,y) = \begin{pmatrix} \tan(xy) \\ \cos(5x-y) \end{pmatrix}$ e.) $f(x,y) = \begin{pmatrix} \sqrt{x^2+y^2} \\ x\sqrt{y} \end{pmatrix}$

6.) Determine the linearization, $L(x)$, for $f(x) = x^2(x-1)$ at $x = 2$. Use $L(x)$ to estimate the value of f at $x = 1.9$.

7.) Determine the linearization, $L(x,y)$, for $f(x,y) = x^2 + 2y^2$ at $(x,y) = (1,-1)$. Use $L(x,y)$ to estimate the value of f at $(x,y) = (1.1,-0.9)$.

8.) Consider the function $f(x,y) = \begin{pmatrix} 2x+6y \\ x^2+2y \end{pmatrix}$. Determine the Jacobi Matrix for each of the points $(1,1), (-2/3,-1)$, and $(-1,1)$. Compute the eigenvalues for each matrix.

9.) Consider the 2x2 linear systems of differential equations given by $X' = AX$, each of which has $(0,0)$ as an equilibrium . For each matrix A below, determine if $(0,0)$ is stable or unstable. Then categorize $(0,0)$ as a sink, source, saddle, stable spiral, unstable spiral, or neutral spiral.

a.) $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ b.) $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ c.) $A = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}$

d.) $A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$ e.) $A = \begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix}$ f.) $A = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$ g.) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

10.) Without explicitly computing the eigenvalues of A, determine whether the real parts of both eigenvalues are negative.

a.) $A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}$ b.) $A = \begin{pmatrix} 0 & 1 \\ -2 & 4 \end{pmatrix}$ c.) $A = \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}$

11.) Determine all equilibria for each of the following 2x2 nonlinear systems of differential equations.

a.)
$$\frac{dx_1}{dt} = x_1^2 - x_2^2$$
$$\frac{dx_2}{dt} = x_1(x_2 - 3)$$

b.)
$$\frac{dx_1}{dt} = 8x_1 - x_1^2 - x_1x_2$$
$$\frac{dx_2}{dt} = 4x_2 - x_1x_2 + x_2^2$$

c.)
$$\frac{dx_1}{dt} = x_1^2x_2 + x_2^2$$
$$\frac{dx_2}{dt} = x_1x_2 - x_1^2 + 2x_1$$

12.) Find all zero isoclines for the systems in problem 11. Plot and label these isoclines using $\frac{dx_1}{dt} = 0$ or $\frac{dx_2}{dt} = 0$ in the x_1x_2-coordinate system. Also label the equilibria.

13.) Set up a table and direction field for the following nonlinear system of differential equations using slopes, direction vectors, and speed for the following points (x_1, x_2) : (0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2).

$$\frac{dx_1}{dt} = x_2^2$$
$$\frac{dx_2}{dt} = x_1x_2$$

14.) Find all equilibria for each system of differential equations, and use the analytical approach to determine the stability of and classify each equilibrium.

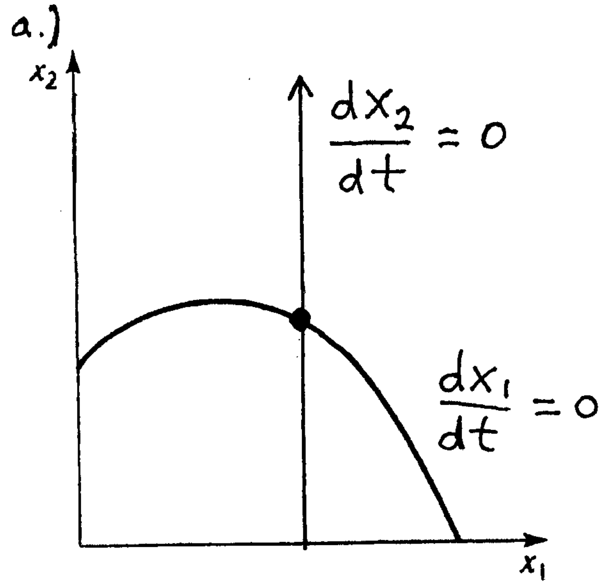
a.)
$$\frac{dx_1}{dt} = x_1 - x_1^2 - x_1x_2$$
$$\frac{dx_2}{dt} = 3x_2 - x_1x_2 - 2x_2^2$$

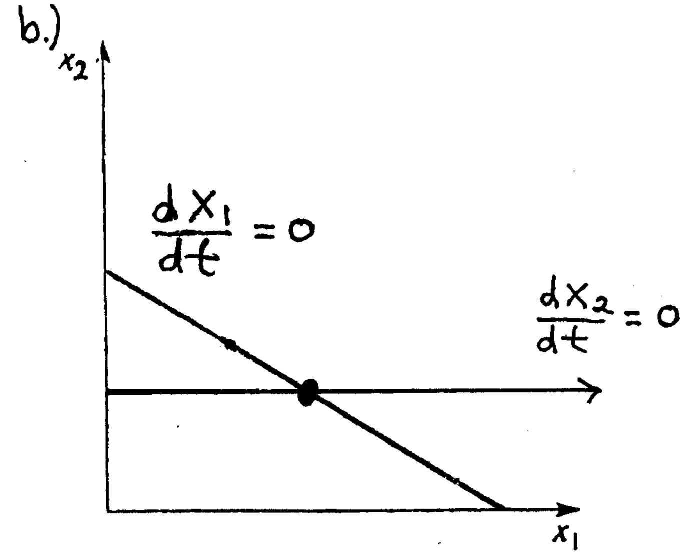
b.)
$$\frac{dx_1}{dt} = x_1x_2 + x_2$$
$$\frac{dx_2}{dt} = -x_1 - x_2$$

15.) Determine the equilibria for the following nonlinear system. Graph the zero isoclines. Use the graphical approach to determine the stability of each equilibrium, if possible.

$$\frac{dx_1}{dt} = x_1 - x_2^2$$
$$\frac{dx_2}{dt} = x_1 - x_2 - 2$$

16.) Consider the system of differential equations given by $\begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2) \\ \frac{dx_2}{dt} = f_2(x_1, x_2) \end{cases}$. For each of the following zero isocline diagrams, assume that both $f_1 > 0$ and $f_2 > 0$ near the origin. Complete the two-dimensional sign chart for f_1 and f_2 using the isocline hopping method (i.e., by assuming the functions change sign when they cross their own isoclines). Then set up a signed Jacobi Matrix to determine (if possible) the stability of the given equilibrium, where the isoclines cross.





17.) Show that $(5, 3)$ is an equilibrium for the following nonlinear system of differential equations. Complete the two-dimensional sign chart for f_1 and f_2 using the isocline hopping method (i.e., by assuming the functions change sign when they cross their own isocline). Then set up a signed Jacobi Matrix to determine (if possible) the stability of the equilibrium, $(5, 3)$.

$$\frac{dx_1}{dt} = 2x_1(3 - x_2)$$
$$\frac{dx_2}{dt} = x_2(x_1 - x_2 - 2)$$

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"I almost had a psychic girlfriend, but she left me before we met." – Steven Wright

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"Nothing will work unless you do." – Maya Angelou