Math 17C (Kouba)

Determining Stability of Equilibria for Nonlinear 2X2 Systems of D.E.'s Using the GRAPHICAL APPROACH

Def: Consider the system of D.E.'s

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2) \end{cases}$$

. a zero isocline is any curve (e.g., function, graph, etc.) which solves

$$f_1(x_1, x_2) = 0 \quad \underline{\text{OR}} \quad f_2(x_1, x_2) = 0 .$$

Note: This is different from a point equilibrium $(x_1, x_2) = (a, b)$, where $f_1(a,b) = 0$ AND $f_2(a,b) = 0$.

Ex: a.) Find, sketch, and label all zero isoclines for

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_2 - x_1 x_2 - x_2^2 = f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = x_1^3 - x_1 x_2 = f_2(x_1, x_2) . \end{cases}$$

$$f_1(x_1, x_2) = 2x_2 - x_1 x_2 - x_2^2 = x_2(2 - x_1 - x_2) = 0$$

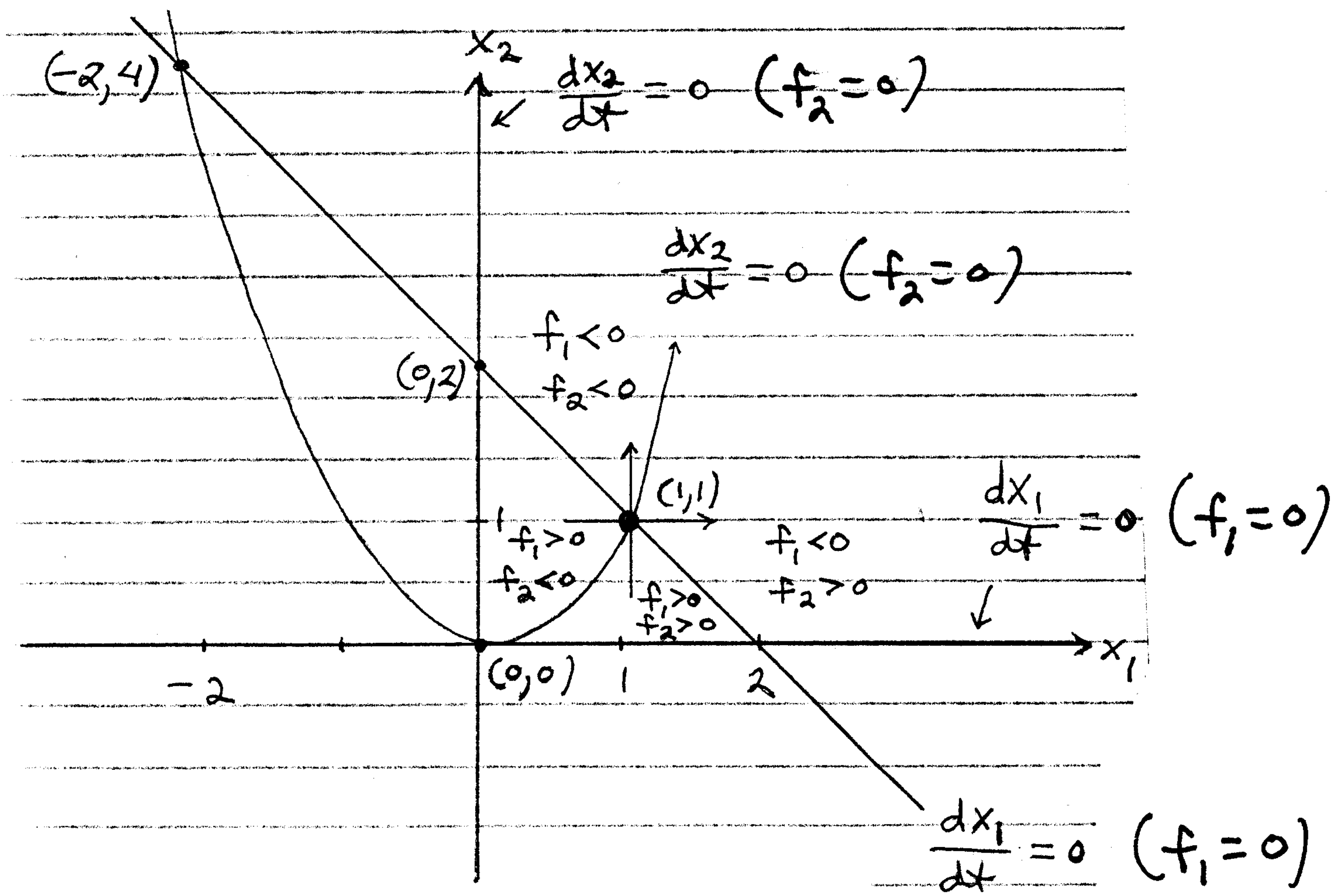
$\rightarrow$ zero isoclines are

$\underline{x_2 = 0}$, $\underline{2 - x_1 - x_2 = 0}$ (both are lines);

$$f_2(x_1, x_2) = x_1^3 - x_1 x_2 = x_1(x_1^2 - x_2) = 0 \rightarrow$$

zero isoclines are

$\underline{x_1 = 0}$, $\underline{x_1^2 - x_2 = 0}$ (line, parabola)



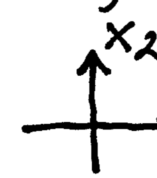
b.) Locate all point equilibria (where $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = 0$ CROSS.):

$(0,0)$, $(0,2)$, $(1,1)$, $(-2,4)$

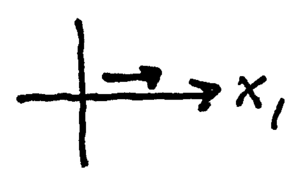
c.) Determine the STABILITY of $(1,1)$ using a GRAPHICAL APPROACH

RECALL: (From earlier this quarter)

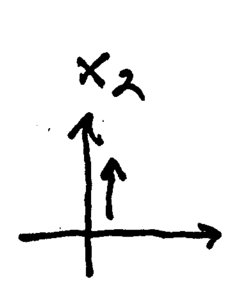
If $z = f(x_1, x_2)$ is a surface in 3D-Space, then



1.) $\frac{\partial f}{\partial x_1}(a,b)$ is the SLOPE of the tangent line at point $(x_1, x_2) = (a,b)$ in the positive x_1-direction:



2.) $\frac{\partial f}{\partial x_2}(a,b)$ is the SLOPE of the tangent line at point $(x_1, x_2) = (a,b)$ in the positive x_2-direction



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i.) Use test points in 4 regions around point (1,1) to determine signs (+ or −) for f_1 and f_2 :

$(1, \tfrac{1}{2})$: $f_1(1, \tfrac{1}{2}) > 0$, $f_2(1, \tfrac{1}{2}) > 0$

$(2, 1)$: $f_1(2,1) < 0$, $f_2(2,1) > 0$

$(1, 2)$: $f_1(1,2) < 0$, $f_2(1,2) < 0$

$(\tfrac{1}{2}, 1)$: $f_1(\tfrac{1}{2}, 1) > 0$, $f_2(\tfrac{1}{2}, 1) < 0$

ii.) Fill in 2-dimensional sign chart for f_1 and f_2 (SEE GRAPH)

iii.) Draw an x_1-arrow and x_2-arrow at point (1,1) and make partial derivative analysis for f_1 and f_2 :

Along x_1-arrow at point (1,1) :

f_1 goes from (+) to (−)
so f_1 is ↓ and $\boxed{\frac{\partial f_1}{\partial x_1} \text{ is } (-)}$;

f_2 goes from (−) to (+)
so f_2 is ↑ and $\boxed{\frac{\partial f_2}{\partial x_1} \text{ is } (+)}$.

<u>along x_2-arrow at point (1,1):</u>

f_1 goes from (+) to (−)
so f_1 is ↓ and $\boxed{\dfrac{\partial f_1}{\partial x_2} \text{ is } (-)}$;

f_2 goes from (+) to (−)
so f_2 is ↓ and $\boxed{\dfrac{\partial f_2}{\partial x_2} \text{ is } (-)}$

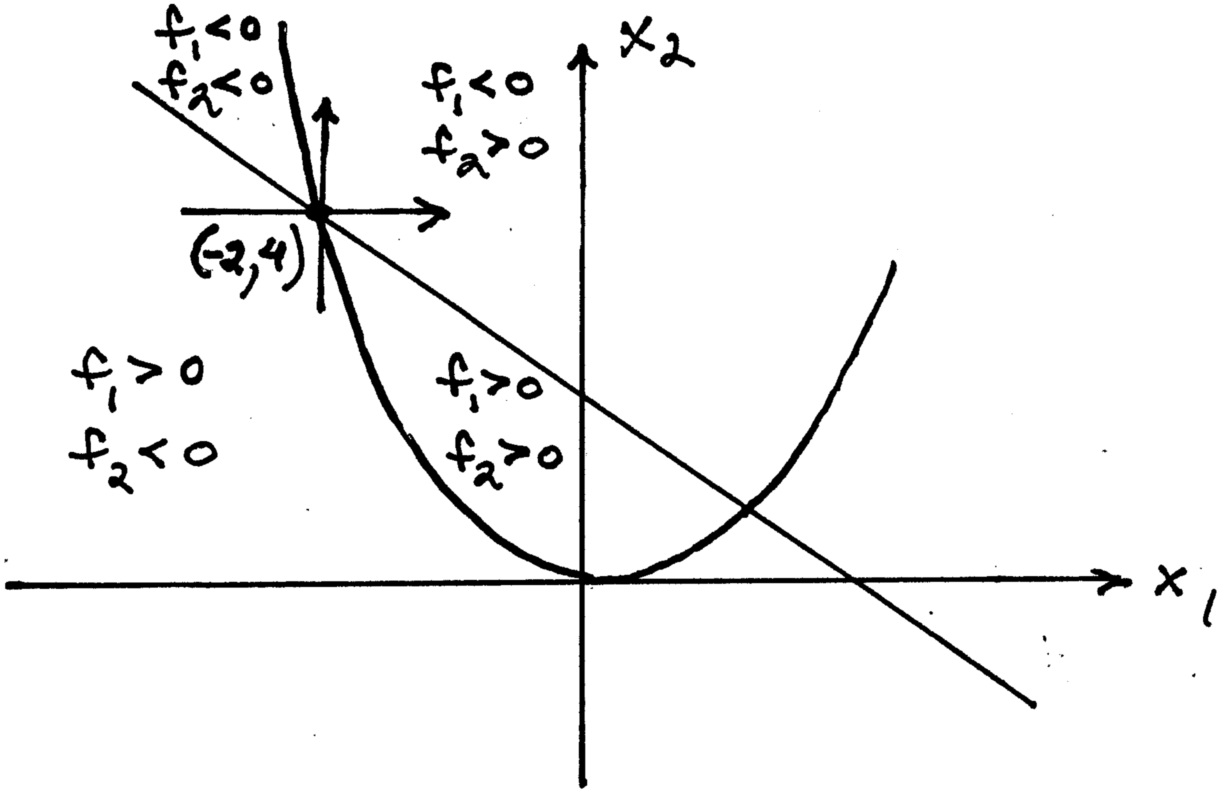
The signed <u>JACOBI MATRIX</u> is now

$$Df(1,1) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(1,1) & \dfrac{\partial f_1}{\partial x_2}(1,1) \\ \dfrac{\partial f_2}{\partial x_1}(1,1) & \dfrac{\partial f_2}{\partial x_2}(1,1) \end{bmatrix} = \begin{bmatrix} - & - \\ + & - \end{bmatrix} = A \, ;$$

then $\operatorname{tr} A = (-) + (-)$ is (−),
and $\det A = (-)(-) - (-)(+)$ is (+).

Thus, by eigenvalue shortcut the <u>real parts</u> of both eigenvalues are <u>negative</u>, and it follows that (1,1) is a <u>STABLE</u> equilibrium.

d.) Determine the STABILITY of (−2, 4) using a GRAPHICAL APPROACH. (optional practice)



<u>along x_1-arrow at point (−2,4):</u>

f_1 goes from (+) to (−)

so f_1 is ↓ and $\boxed{\frac{\partial f_1}{\partial x_1} \text{ is } (-)}$;

f_2 goes from (−) to (+)

so f_2 is ↑ and $\boxed{\frac{\partial f_2}{\partial x_1} \text{ is } (+)}$.

along x_2-arrow at point $(-2,4)$:

f_1 goes from $(+)$ to $(-)$
so f_1 is $\downarrow$ and $\boxed{\dfrac{\partial f_1}{\partial x_2} \text{ is } (-)}$;

f_2 goes from $(-)$ to $(+)$
so f_2 is $\uparrow$ and $\boxed{\dfrac{\partial f_2}{\partial x_2} \text{ is } (+)}$

The signed JACOBI MATRIX is now

$$Df(-2,4) = \begin{bmatrix} - & - \\ + & + \end{bmatrix} = A; \text{ then}$$

$\text{tr}\, A = (-) + (+) = ?!\star$ and
$\det A = (-)(+) - (-)(+) = ?!\star$ so
NO CONCLUSION can be made about the stability of point $(-2,4)$ using this graphical approach.

<u>Note</u>: The graphical approach is NOT always conclusive:

<u>Ex</u>: 1.) If $A = \begin{bmatrix} - & - \\ - & + \end{bmatrix}$, then

$\operatorname{tr} A = (-) + (+)$ is inconclusive.

2.) If $A = \begin{bmatrix} - & + \\ + & - \end{bmatrix}$, then

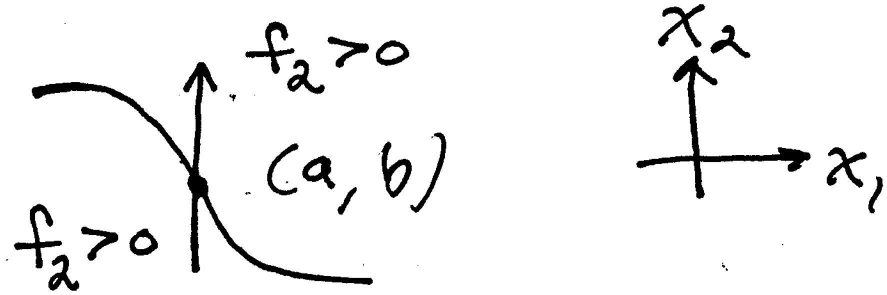
$\operatorname{tr} A = (-) + (-)$ is $(-)$, but

$\det A = (-)(-) - (+)(+) = (+) - (+)$

is <u>inconclusive</u>.

<u>Note</u>: If

<u>Ex</u>:



then we conclude $\dfrac{\partial f_2}{\partial x_2} = 0$.

If $Df(a,b) = \begin{bmatrix} - & - \\ + & 0 \end{bmatrix} = A$, then

$\operatorname{tr} A = (-) + (0) = (-)$ and

$\det A = (-)(0) - (-)(+)$ is $(+)$. Thus by eigenvalue shortcut the <u>real parts</u> of both eigenvalues are <u>negative</u>, and it follows that (a,b) is STABLE equilibrium.