Math 17C
Koulba
Parametric Equations

Parametric Equations

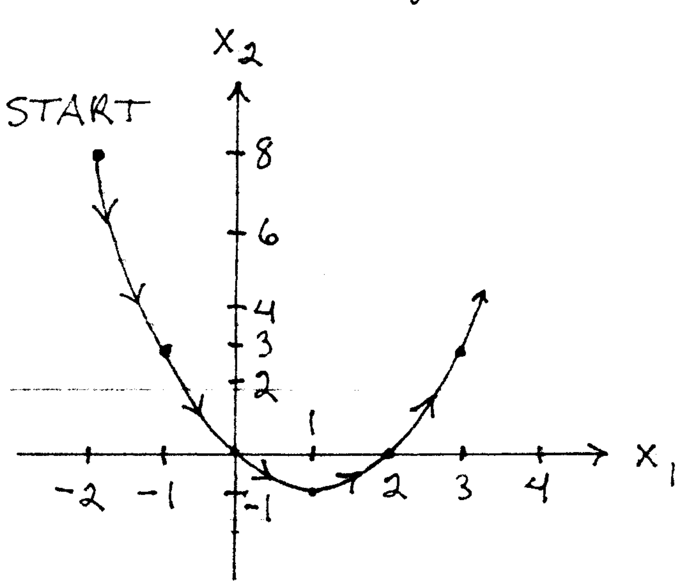
Ex: The position (x_1, x_2) of a particle at time t is given parametrically by

$$\begin{cases} x_1 = t-2 \\ x_2 = t^2-6t+8 \quad \text{for } t \geq 0 . \end{cases}$$

1.) Find the particle's position when $t = 0, 1, 2, 3, 4, 5$:

t:	0	1	2	3	4	5
x_1:	-2	-1	0	1	2	3
x_2:	8	3	0	-1	0	3

2.) Sketch the path and indicate the direction of motion :



3.) Remove parameter t and write the equation of the path in terms of x_1 and x_2 only :

$$\begin{cases} x_1 = t-2 \\ x_2 = t^2-6t+8 \end{cases} \rightarrow t = x_1+2 \rightarrow \text{(SUB)} \rightarrow$$

$$x_2 = (x_1+2)^2-6(x_1+2)+8 \rightarrow$$

$$x_2 = x_1^2+4x_1+4-6x_1-12+8 \rightarrow$$

$$\boxed{x_2 = x_1^2-2x_1} \quad \text{(parabola)}$$

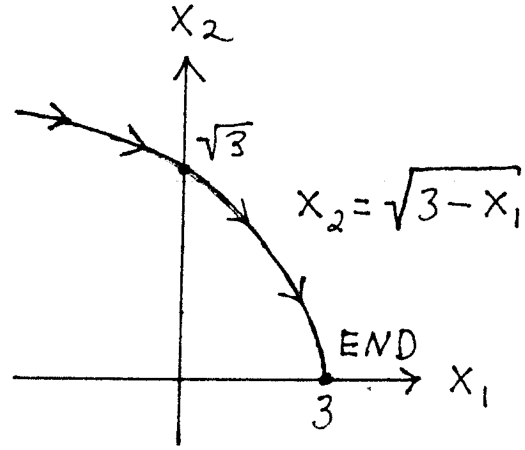
<u>Ex</u>: For each parametric representation write an equation of the path in terms of x_1 and x_2 only; sketch the path in the x_1x_2-plane; indicate the direction of motion:

1.) $\begin{cases} x_1 = t+1 \\ x_2 = \sqrt{2-t} \end{cases}$ for $t \leq 2$

$\rightarrow t = x_1-1 \rightarrow \text{(SUB)} \rightarrow$

$x_2 = \sqrt{2-(x_1-1)} = \sqrt{3-x_1} \rightarrow$

$\underline{x_2 = \sqrt{3-x_1}}$

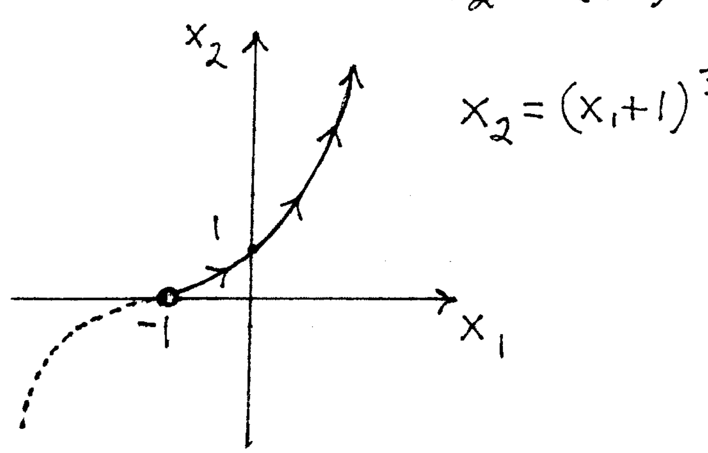


2.) $\begin{cases} x_1 = e^t-1 \\ x_2 = e^{3t} \end{cases}$ for $-\infty < t < \infty$ $\rightarrow \begin{cases} x_1 = e^t-1 \\ x_2 = (e^t)^3 \end{cases}$

$\rightarrow \begin{cases} e^t = x_1+1 \\ x_2 = (e^t)^3 \end{cases}$

$\rightarrow \underline{x_2 = (x_1+1)^3}$

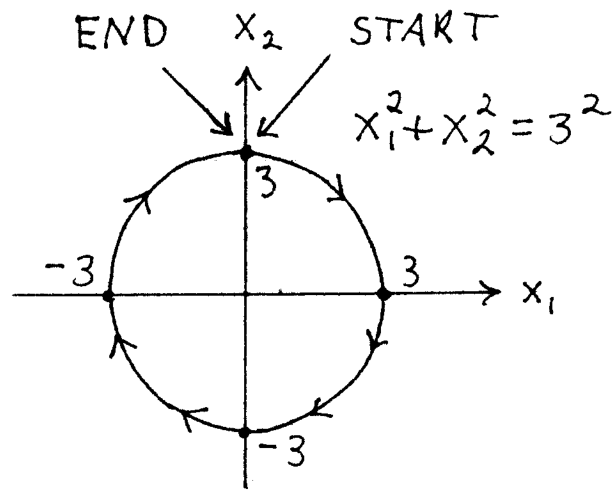
(cubic)



3.) $\begin{cases} x_1 = 3\sin t \\ x_2 = 3\cos t \end{cases}$ for $0 \le t \le 2\pi$ $\longrightarrow$

$$\begin{aligned} x_1^2 + x_2^2 &= (3\sin t)^2 + (3\cos t)^2 \\ &= 9\sin^2 t + 9\cos^2 t \\ &= 9(\sin^2 t + \cos^2 t) \\ &= 9(1) \longrightarrow \end{aligned}$$

$\underline{x_1^2 + x_2^2 = 9}$ (circle)



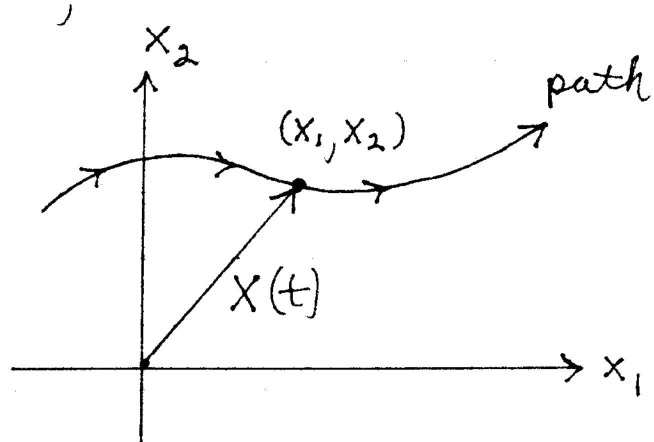
Tangent Vectors and Speed along a Parametric Path

Consider the path given parametrically by $\begin{cases} x_1 = f(t) \\ x_2 = g(t) \end{cases}$ for $-\infty < t < \infty$.

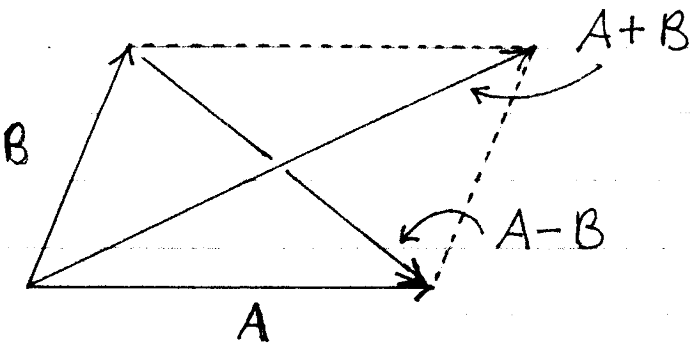
Let vector function

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} ,$$

so that the tip of the vector traces out a path in the x_1x_2-plane.



Recall :



The derivative of vector function $X(t)$ is

$$X'(t) = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

$$= \lim_{h \to 0} \frac{\begin{bmatrix} f(t+h) \\ g(t+h) \end{bmatrix} - \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}}{h}$$

$$= \lim_{h \to 0} \begin{bmatrix} \dfrac{f(t+h) - f(t)}{h} \\ \dfrac{g(t+h) - g(t)}{h} \end{bmatrix} = \begin{bmatrix} f'(t) \\ g'(t) \end{bmatrix} \quad ;$$

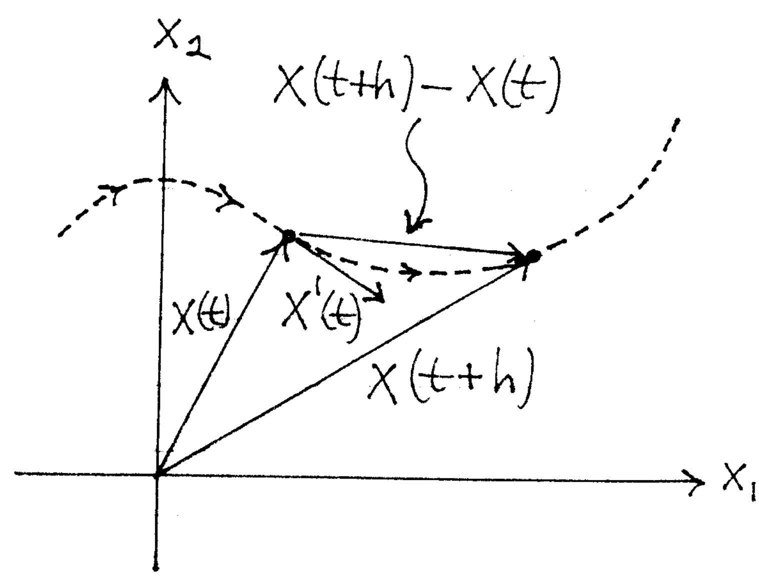
and

$$X'(t) = \lim_{h \to 0} \frac{1}{h} \left(X(t+h) - X(t) \right)$$

is a vector, called a velocity vector, which is

i.) tangent to the path of the particle

and ii.) points in the direction of motion



Speed of Motion Along a Path, $X(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

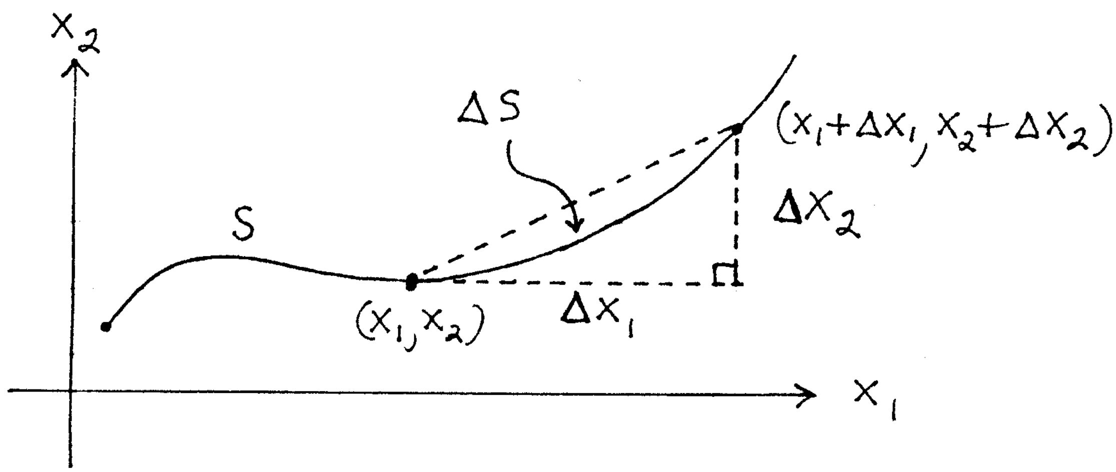
Consider the following changes:

time : $t \to t + \Delta t$

$x_1 \to x_1 + \Delta x_1$

$x_2 \to x_2 + \Delta x_2$

arc length : $s \to s + \Delta s$



Then $(\Delta s)^2 \approx (\Delta x_1)^2 + (\Delta x_2)^2 \longrightarrow$

$$\frac{(\Delta s)^2}{(\Delta t)^2} \approx \frac{(\Delta x_1)^2}{(\Delta t)^2} + \frac{(\Delta x_2)^2}{(\Delta t)^2} \longrightarrow$$

$$\left(\frac{\Delta s}{\Delta t}\right)^2 \approx \left(\frac{\Delta x_1}{\Delta t}\right)^2 + \left(\frac{\Delta x_2}{\Delta t}\right)^2 \longrightarrow$$

(Let $\Delta t \to 0$ as a limit.)

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 \longrightarrow$$

SPEED is $\boxed{\frac{ds}{dt} = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2} = |X'(t)|}$

(SPEED = magnitude of velocity vector)

Ex: Consider the path given parametrically by $\begin{cases} x_1 = t \\ x_2 = t^2 - 2t \end{cases}$ for $t \geq 0$.

a.) Plot the path in the x_1x_2-plane.

b.) Find a vector tangent to the path, pointing in the direction of motion, and the speed of motion when $t = 0, 1,$ and 3:

$$X(t) = \begin{bmatrix} t \\ t^2 - 2t \end{bmatrix}, \quad X'(t) = \begin{bmatrix} 1 \\ 2t - 2 \end{bmatrix}, \quad \frac{ds}{dt} = |X'(t)| ;$$

$$t = 0: \quad X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X'(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \frac{ds}{dt} = \sqrt{5},$$

$$t = 1: \quad X(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad X'(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \frac{ds}{dt} = 1,$$

$$t = 3: \quad X(3) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad X'(3) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \frac{ds}{dt} = \sqrt{17}$$

$$\begin{cases} x_1 = t \\ x_2 = t^2 - 2t \end{cases} \longrightarrow x_2 = x_1^2 - 2x_1$$