Lecture 14

Math 17C
Sec. 11.4
HW # 10

Applications of Nonlinear Systems

Nonlinear systems of D.E.'s can be used to describe population changes between competing species, predator/prey dynamics, Neuron activity of the nervous system, enzyme reactions in biochemistry, mixture problems, etc.

Lotka-Voltera (Ukraine, Italy in early 1900's) Predator-Prey Model —

Let $N(t)$: prey population at time t
$P(t)$: predator population at time t;

We will assume $N(t) \xrightarrow{exp.} \infty$ as $t \to \infty$ in the absence of predators, i.e.,

(I) $\frac{dN}{dt} = rN(t)$ (for $r > 0$);

we will assume $P(t) \xrightarrow{exp.} 0$ as $t \to \infty$ in the absence of prey, i.e.,

(II) $\frac{dP}{dt} = -d\,P(t)$ (for $d > 0$).

However, when predator and prey are in the same habitat, we can expect the following changes:

1. The number of encounters between predator and prey is proportional to the product of their populations.
   a. Each encounter promotes the growth of the predator.
   b. Each encounter inhibits the growth of the prey.
2. adjust (I) by
   $-a\,P(t)N(t)$, where
   a is a measure of the attack rate.
3. adjust (II) by $b(a\,P(t)N(t))$,

where b is a measure of how effectively the attack rate influences reproductive rates of the predator.

Thus, the model becomes:

$$(*)\quad \begin{cases} \dfrac{dN}{dt} = r\,N(t) - a\,P(t)\,N(t) \\[2ex] \dfrac{dP}{dt} = b\,a\,P(t)\,N(t) - d\,P(t)\,. \end{cases}$$

<u>Note</u>: This model is solvable since the variables separate:

$$\begin{cases} \dfrac{dN}{dt} = rN - aPN = N(r-aP) \\[1ex] \dfrac{dP}{dt} = baPN - dP = P(baN-d) \end{cases}$$

$$\rightarrow \quad \frac{dP}{dN} = \frac{\frac{dP}{dt}}{\frac{dN}{dt}} = \frac{P(baN-d)}{N(r-aP)} \longrightarrow$$

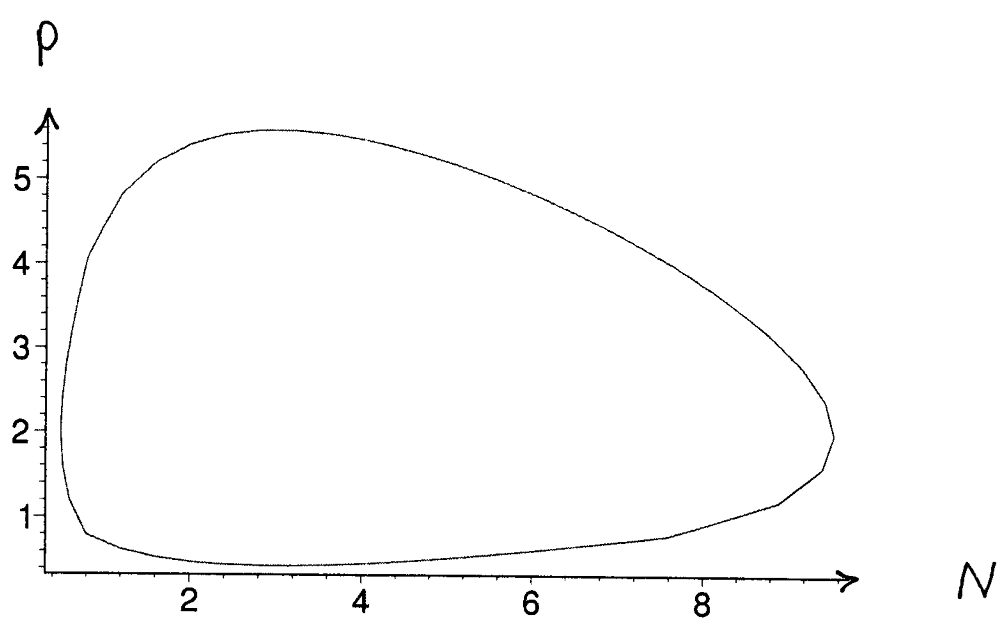
$$\int \frac{r-aP}{P}\,dP = \int \frac{baN-d}{N}\,dN \longrightarrow$$

$$\int \left[\frac{r}{P} - a\right] dP = \int \left[ba - \frac{d}{N}\right] dN \longrightarrow$$

$$\boxed{r \ln P - aP = baN - d \ln N + c} \; ;$$

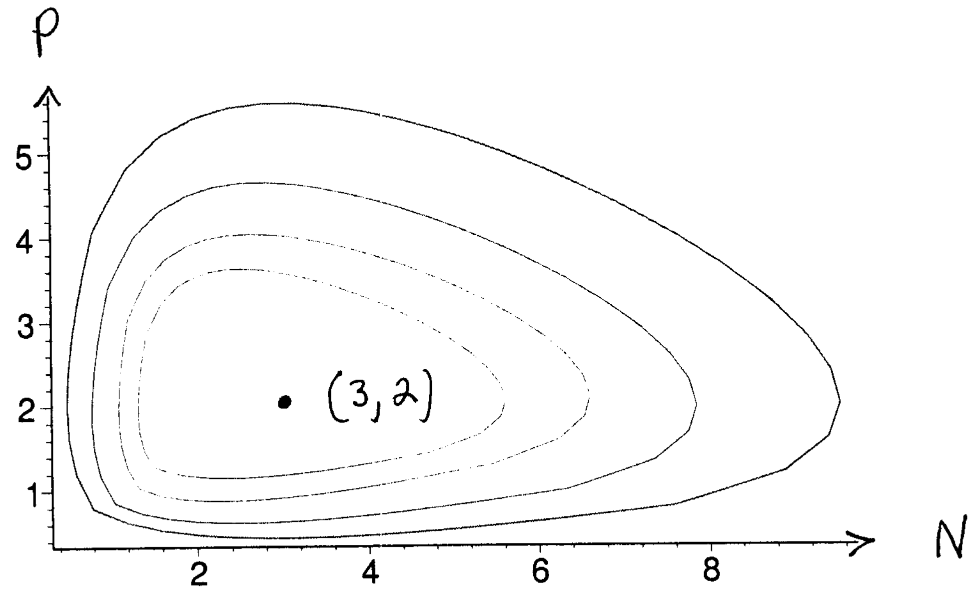
Ex: If $a = 2$, $b = 1/2$, $r = 4$, $d = 3$, and $c = -4$, then graph of

$$4 \ln P - 2P = N - 3 \ln N - 4 \quad \text{is}$$



$$\begin{cases} \dfrac{dN}{dt} = 4N - 2PN = 2N(2 - P) \\[2ex] \dfrac{dP}{dt} = PN - 3P = P(N-3) \end{cases}$$

Note: Depending on initial conditions $N(0)$ and $P(0)$, we get closed, concentric solution curves about the equilibrium $(3,2)$:



Note that equilibria for this system are $(0,0)$ and $(3,2)$. Use eigenvalues to determine stability:

$$f(N,P) = \begin{bmatrix} 4N - 2PN \\ PN - 3P \end{bmatrix} \longrightarrow \text{Jacobi Matrix}$$

$$Df(N,P) = \begin{bmatrix} 4-2P & -2N \\ P & N-3 \end{bmatrix}, \text{ ~~then~~}$$

a.) $Df(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} = A \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix}$

$= (4-\lambda)(-3-\lambda) = 0 \rightarrow \lambda_1 = 4,\ \lambda_2 = -3,$
so $(0,0)$ is <u>unstable</u> equilibrium

b.) $Df(3,2) = \begin{bmatrix} 0 & -6 \\ 2 & 0 \end{bmatrix} = A \rightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 0-\lambda & -6 \\ 2 & 0-\lambda \end{bmatrix}$

$= \lambda^2 + 12 = 0 \rightarrow \lambda^2 = -12 \rightarrow$
$\lambda = \pm\sqrt{12}\, i \rightarrow \lambda_1 = \sqrt{12}\, i,\ \lambda_2 = -\sqrt{12}\, i$;
since eigenvalues are purely imaginary, this analysis does not lead to a conclusion about the stability of $(3,2)$.