Math 17C
Kouba
Exam 1 (Practice)   KEY

Please PRINT your name here : _____

Your Exam ID Number _____

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 8:50 a.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. Make sure that you have 7 pages including the cover page.

1

1.) Consider the function given by $z = x^2 + y^2 - 4$ and its graph in 3D-space.

a.) (4 pts.) Determine all possible intercepts for this equation.

$X = 0, Y = 0 \implies Z = -4$,

$X = 0, Z = 0 \implies Y^2 = 4 \implies Y = \pm 2$,

$Y = 0, Z = 0 \implies X^2 = 4 \implies X = \pm 2$.

b.) (4 pts.) Determine and sketch the xy-trace, xz-trace, and yz-trace.

$Z = 0: X^2 + Y^2 = 4$
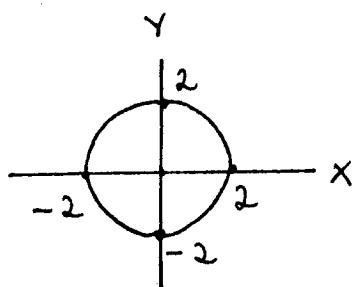
$Y = 0: Z = X^2 - 4$
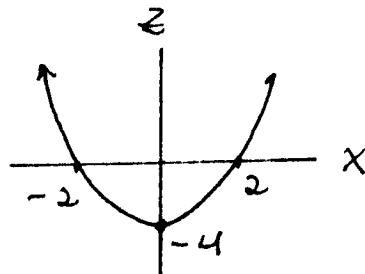
$X = 0: Z = Y^2 - 4$
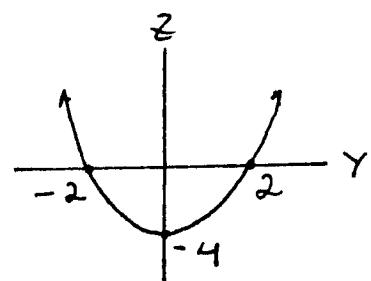
c.) (4 pts.) Sketch level curves on the same axes for $z = -3, 0,$ and 5.

| z-value	level curve
-3	$X^2 + Y^2 = 1$
0	$X^2 + Y^2 = 4$
5	$X^2 + Y^2 = 9$
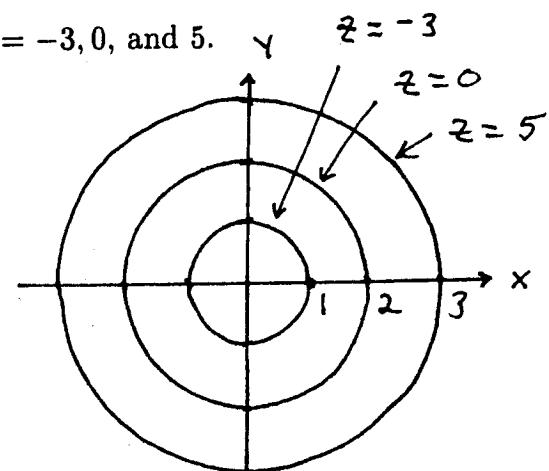
d.) (4 pts.) Sketch the surface in 3D-space.
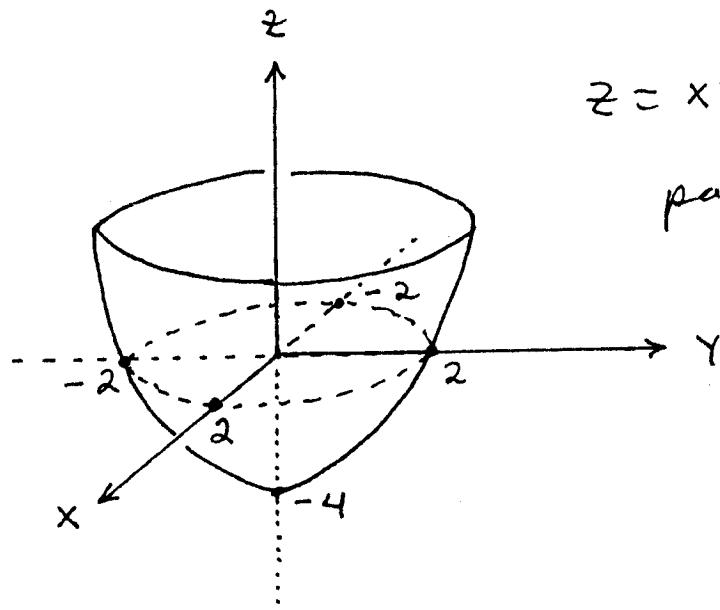
$Z = X^2 + Y^2 - 4$

paraboloid

2.) (9 pts.) Let $z = 3x + \ln(x^2 + y)$. Compute the partial derivatives z_x, z_y, and z_{xx}. DO NOT SIMPLIFY your answers.

$$z_x = 3 + \frac{2X}{X^2 + Y} \quad , \quad z_y = \frac{1}{X^2 + Y} \quad ,$$

$$z_{xx} = \frac{(X^2 + Y)(2) - 2X(2X)}{(X^2 + Y)^2}$$

3.) (11 pts.) Find and classify (relative maximum value, relative minimum value, or saddle point) each critical point for the following function :

$$f(x, y) = x^3 - y^3 + 3xy$$

$$f_x = 3X^2 + 3Y = 3(X^2 + Y) = 0 \rightarrow \boxed{Y = -X^2} \; ;$$

$$f_y = -3Y^2 + 3X = 3(-Y^2 + X) = 0 \rightarrow \boxed{X = Y^2} \; ;$$

substitute $\Rightarrow Y = -(X^2) = -(Y^2)^2 = -Y^4 \Rightarrow$

$Y^4 + Y = Y(Y^3 + 1) = 0 \Rightarrow Y = 0$ or $Y = -1$; if $Y = 0$, then

$X = 0$, so $\boxed{(0,0)}$ is critical point ; if $Y = -1$, then

$X = 1$, so $\boxed{(1,-1)}$ is critical point ; then

$f_{xx} = 6X$, $f_{yy} = -6Y$, $f_{xy} = 3$;

check $(0,0)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (0)(0) - (3)^2 = -9 < 0$

so $(0,0)$ determines a <u>saddle point</u>.

check $(1,-1)$: $D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (6)(6) - (3)^2 = 27 > 0$

and $f_{xx} = 6 > 0$ so $(1,-1)$ determines a

relative <u>minimum value</u> of $f(1,-1) = -1$.

4.) (9 pts.) Evaluate the following limit : $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{x+y+1}}{x+y}$

$$\overset{"\frac{0}{0}"}{=} \lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{X+Y+1}}{X+Y} \cdot \frac{1+\sqrt{X+Y+1}}{1+\sqrt{X+Y+1}}$$

$$= \lim_{(x,y)\to(1,-1)} \frac{1-(X+Y+1)}{(X+Y)(1+\sqrt{X+Y+1})}$$

$$= \lim_{(x,y)\to(1,-1)} \frac{-(X+Y)}{(X+Y)(1+\sqrt{X+Y+1})}$$

$$= \frac{-1}{1+\sqrt{1-1+1}} = \frac{-1}{1+1} = \frac{-1}{2}$$

5.) (9 pts.) Verify that the following limit does not exist : $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

<u>along path</u> $Y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{XY^3}{X^2+Y^6} = \lim_{(x,y)\to(0,0)} \frac{0}{X^2} = \lim_{X\to0} 0 = \boxed{0} \ ;$$

<u>along path</u> $X=Y^3$:

$$\lim_{(x,y)\to(0,0)} \frac{XY^3}{X^2+Y^6} = \lim_{(x,y)\to(0,0)} \frac{Y^3 Y^3}{(Y^3)^2+Y^6}$$

$$= \lim_{Y\to0} \frac{Y^6}{Y^6+Y^6} = \lim_{Y\to0} \frac{Y^6}{2Y^6}$$

$$= \lim_{Y\to0} \frac{1}{2} = \boxed{\frac{1}{2}} \ ; \quad \text{so} \lim_{(x,y)\to(0,0)} \frac{XY^3}{X^2+Y^6} \ \text{DNE.}$$

6.) Let $f(x,y) = 4 - \sqrt{y - x^2}$.

    a.) (5 pts.) Determine and sketch the domain of f in 2D-space.

$Y - X^2 \geq 0 \Rightarrow Y \geq X^2$, so

Domain is set of all points (X,Y) on or above the graph of $Y = X^2$.
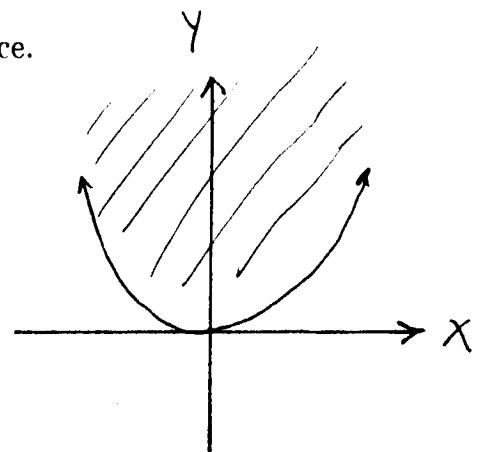
    b.) (5 pts.) State the range of f. BRIEFLY explain how you get your answer.

along path $X = 0$, $Z = \sqrt{Y - X^2} = \sqrt{Y}$ goes from 0 to $+\infty$. Thus, the Range $Z = 4 - \sqrt{Y - X^2}$ is $-\infty < Z \leq 4$.

7.) (9 pts.) Give a precise ϵ, δ-proof that $\displaystyle\lim_{(x,y)\to(1,0)} (xy + x) = 1$:

Let $\epsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + Y^2} < \delta$

then $|(XY + x) - 1| < \epsilon$. Then

$|XY + x - 1| = |(x-1)Y + Y + (x-1)|$

$\Delta \text{ In.} \leq |x-1||Y| + |Y| + |x-1|$

$= \sqrt{(x-1)^2} \cdot \sqrt{Y^2} + \sqrt{Y^2} + \sqrt{(x-1)^2}$

$\leq \sqrt{(x-1)^2 + Y^2} \cdot \sqrt{(x-1)^2 + Y^2} + \sqrt{(x-1)^2 + Y^2} + \sqrt{(x-1)^2 + Y^2}$

$= \left(\sqrt{(x-1)^2 + Y^2}\right)^2 + 2\sqrt{(x-1)^2 + Y^2}$

assume

$\delta \leq 1 \leq \sqrt{(x-1)^2 + Y^2} + 2\sqrt{(x-1)^2 + Y^2} = 3\sqrt{(x-1)^2 + Y^2} < \epsilon$

iff $\sqrt{(x-1)^2 + Y^2} < \epsilon/3$. Now choose

$\delta = \min\left\{ \epsilon/3, 1 \right\}$ and the result follows.

QED

8.) (9 pts.) Consider the surface given by $z = xy^2 - 3y$ and point $P = (1, -1, 4)$ on this surface. Determine an equation for the plane tangent to this surface at point P.

$z_x = y^2$ and $x = 1, y = -1 \rightarrow z_x = (-1)^2 = 1$,

$z_y = 2xy - 3$ and $x = 1, y = -1 \rightarrow z_y = 2(1)(-1) - 3 = -5$;

and $z = (1)(-1)^2 - 3(-1) = 4$ so tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \rightarrow$$

$$z = 4 + (1)(x - 1) + (-5)(y - (-1))$$

$$= 4 + x - 1 - 5y - 5 \rightarrow$$

$$z = x - 5y - 2$$

9.) (9 pts.) Consider the function $f(x, y) = x^2 - y^2$, the point $P = (2, -1)$, and vector $\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Compute the directional derivative of f in the direction of $\vec{v}$ at point P.

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix} \text{ so } \nabla f(2, -1) = \begin{bmatrix} 2(2) \\ -2(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix};$$

$$\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \text{ so } |\vec{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5, \text{ so}$$

unit vector is $\vec{u} = \frac{1}{5}\vec{v} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix};$ then

$$D_{\vec{u}} f(2, -1) = \nabla f(2, -1) \cdot \vec{u}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} = \frac{12}{5} - \frac{8}{5} = \frac{4}{5}$$

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10.) (9 pts.) Find the linearization $L(x)$ of $f(x,y) = \begin{pmatrix} 3x + y^2 \\ x^2y \end{pmatrix}$ at $(1, -1)$.

$$Df(x,y) = \begin{bmatrix} 3 & 2y \\ 2xy & x^2 \end{bmatrix} \rightarrow Df(1,-1) = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} ; \text{ then}$$

$$L(x,y) = f(a,b) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$= f(1,-1) + Df(1,-1)\begin{bmatrix} x-1 \\ y+1 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 3x-3 - 2y-2 \\ -2x+2 + y+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 3x-2y-5 \\ -2x+y+3 \end{bmatrix} = \begin{bmatrix} 3x-2y-1 \\ -2x+y+2 \end{bmatrix}$$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OP-TIONAL.

1.) Consider the graph of $z = \sqrt{x}$ in the xz-plane. Find an equation for the surface created by revolving this graph about the z-axis.

circle

$z = \sqrt{x}$
or
$x = z^2$
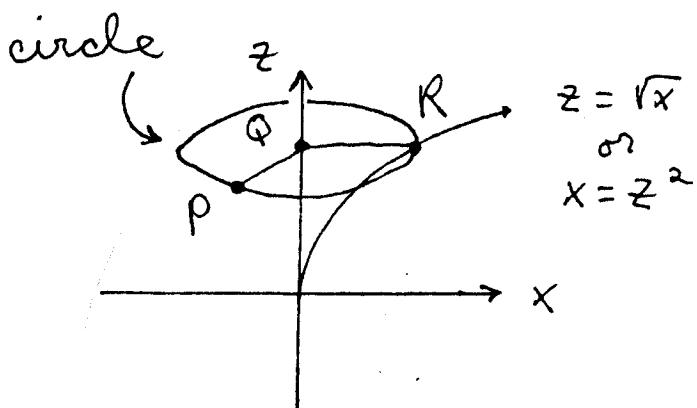
Points $P = (x, y, z)$,

$Q = (0, 0, z)$, $R = (z^2, 0, z)$,

then distances

$\overline{PQ}^2 = \overline{QR}^2 \Rightarrow$

$x^2 + y^2 = (z^2)^2$

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