Math 21A
Kouba
The Squeeze Principle

A useful tool in the the determination of limits of functions is the Squeeze Principle. It can be used when ambiguous or indeterminate forms arise, often times with problems involving oscillating trigonometric functions, and more common algebraic means may fail.

The Squeeze Principle : Let f, g, and h be functions. If

$$g(x) \leq f(x) \leq h(x) \qquad \textit{and} \qquad \lim_{x \to a} g(x) = L = \lim_{x \to a} h(x) \ ,$$

then

$$\lim_{x \to a} f(x) = L \ .$$

Remark : The term "a" can be a finite number or $\pm\infty$.

Example : Evaluate $\displaystyle \lim_{x \to -\infty} \frac{x + \cos^2 4x}{x^2 + 13}$. (Note that $\cos^2 4x$ oscillates.)

Begin with the well known fact that $-1 \leq \cos 4x \leq 1$. Then

$$0 \leq \cos^2 4x \leq 1^2 \quad \longrightarrow$$

$$x \leq x + \cos^2 4x \leq x + 1 \quad \longrightarrow$$

$$\frac{x}{x^2 + 13} \leq \frac{x + \cos^2 4x}{x^2 + 13} \leq \frac{x+1}{x^2 + 13} \ .$$

Since $\displaystyle \lim_{x \to -\infty} \frac{x}{x^2 + 13} = \lim_{x \to -\infty} \frac{x}{x^2 + 13} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to -\infty} \frac{1/x}{1 + 13/x^2} = \frac{0}{1+0} = 0$

and $\displaystyle \lim_{x \to -\infty} \frac{x+1}{x^2 + 13} = \lim_{x \to -\infty} \frac{x+1}{x^2 + 13} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to -\infty} \frac{1/x + 1/x^2}{1 + 13/x^2} = \frac{0+0}{1+0} = 0$

it follows from the Squeeze Principle that

$$\lim_{x \to -\infty} \frac{x + \cos^2 4x}{x^2 + 13} = 0 \ .$$