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Item A

Math 21A **Fall 1993** **Final Exam**

1. Find each of the following limits, showing your work carefully. (18 points)

   a) $\lim_{x\to 3} \dfrac{|x^2 - 9| + \sin(2x - 6)}{2x^2 - 7x + 3}$

   b) $\lim_{x\to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x}$

   c) $\lim_{x\to \infty} \left(\sqrt{2x^2 + 3x - 4} - \sqrt{2x^2 - 3x + 4}\right)$

2. a) State the definition of the derivative of $f(x)$. (5 points)

   b) Use the definition of the derivative to find $f'(x)$ for $f(x) = \sqrt{x + 1} - x$. (10 points)

3. Differentiate the following functions. (Do <u>not</u> simplify your answers.) (24 points)

   a) $f(x) = \sqrt{1 + (x^2 + 4)^6}$

   b) $f(x) = 2e^{1/x}\sin(x^3)$

   c) $f(x) = \ln(\sin(\cos x))$

   d) $f(x) = \dfrac{\arctan x}{\ln(3x + 1)}$

4. a) State the precise definition of $\lim_{x\to a} f(x) = L$. (5 points)

   b) Use the precise definition of the limit to prove that

   $$\lim_{x\to 2} (2x^2 + 5) = 13.$$ (13 points)

5. a) State the Mean–Value Theorem. (5 points)

b) Show that the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ satisfies the hypotheses of the Mean–Value Theorem on the interval $[-\pi/2, \pi/2]$. (11 points)

c) What conclusion follows from the Mean–Value Theorem in this case? (4 points)

6. A rock is thrown straight down from the top of a building with an initial velocity of 16 ft/sec (taking the positive direction downward). Answer the following questions, assuming that the acceleration of the rock is always 32 ft/sec^2 and the building is 64 feet tall.

a) Find a function $y = f(t)$ that gives the position of the rock after t seconds, if $y = 0$ at the top of the building. (9 points)

b) How long does it take the rock to go halfway down to the ground? (5 points)

c) How fast is the rock travelling when it is halfway down to the ground? (3 points)

7. Consider the function $f(x) = \frac{1}{7}x^7 + \frac{7}{4}x^4 - 8x$. Its first and second derivatives are $f'(x) = x^6 + 7x^3 - 8$ and $f''(x) = 6x^5 + 21x^2$.

a) Find the relative extrema of f, and show whether each extremum is a maximum or a minimum. (8 points)

b) Describe the intervals where f is concave up or concave down. (6 points)

c) Use the information in parts (a) and (b) to sketch the graph of f. (6 points)

8. A company manufactures cookie cutters in the shape of a circular sector. These cutters enclose an area of 6 in^2 and are made from three strips of metal: two straight and one circular. The straight pieces cost 2 cents/inch while the curved one costs 3 cents/inch, and it costs 5 cents to weld the three pieces together.

Find the values of r and s which the cutter should have to minimize the manufacturing cost, and show that your answer gives a minimum. (18 points)

[Recall that the area of a circular sector is proportional to its central angle θ.]

9. Let f(x) be a function whose second derivative is given by

$$f''(x) = e^{-x} + x^3 - 3x + 1.$$

Show that f(x) has at least two inflection points. <u>Carefully explain your reasoning</u>! (15 points)

10. a) Use implicit differentiation to calculate the slope at a general point (x, y) of the curve defined by $7x^2 + 2xy + y^2 = 4$. (8 points)

b) Find the equation of the tangent line to the curve in part (a) at the point (0, 2). (5 points)

c) Use implicit differentiation to find the derivative of $y = \arccos x$. (7 points)

11. In each case, either give an example of a function f with the stated properties <u>or</u> explain why there is no such function: (15 points)

a) f is continuous everywhere, but is not differentiable at $x = 0$.

b) f is differentiable everywhere, but is not continuous at $x = 0$.

c) f is continuous everywhere, but f has an antiderivative which is not continuous at $x = 0$.

Math 21A **Final Exam** **Fall 1992**

1. Differentiate each of the following with respect to x. Do NOT simplify your answers.

[9] (a) $f(x) = \sec^2 3x - \dfrac{\cos x}{x^{1/5}}$

[7] (b) $f(x) = e^{x^2} \sin^{-1}(1 + x)$ [note: $\sin^{-1}(1 + x) = \arcsin(1 + x)$]

[9] (c) $y = \ln\sqrt{1 + \sqrt{\ln x}}$

[15] 2. Find $\frac{dy}{dx}$ if x and y are related implicitly by $\tan^{-1}(2y) = xy$.
[note: $\tan^{-1}(2y) = \arcsin(2y)$]

3.

[5] (a) State the definition of the derivative $f'(x)$ of a function $f(x)$.

[10] (b) Use the definition of the derivative to calculate $f'(x)$ for $f(x) = \dfrac{1}{\sqrt{x}}$.

[13] 4. Let

$$f(x) = \begin{cases} 1 - x^2, & \text{if } x \le 1 \\ \frac{1}{2}\tan(\pi x), & \text{if } x > 1 . \end{cases}$$

Check the continuity of f at $x = 1$.

[20] 5. A rectangle of perimeter p is rotated about one of its sides to form a cylinder. Of all such possible rectangles, find the dimensions of the rectangle that generates the cylinder of maximum volume.

6. For each of the following, either find the desired limit or state that "no limit exists". Show all work needed to obtain your answers.

[6] (a) $\lim_{x \to -\infty} \frac{\sqrt{9x^2 + 2x + 5}}{4x}$

[7] (b) $\lim_{x \to 0} \frac{x^2 + \sin 2x}{3x}$

[9] (c) $\lim_{x \to 0^+} x^{\sqrt{x}}$

[10] 7. Use differentials to estimate $e^{0.1}$.

8. Consider the function $f(x) = \sin 2x + 2\sin x$, which has derivatives

$f'(x) = 2\cos 2x + 2\cos x$, and $f''(x) = -4\sin 2x - 2\sin x$.

Justify your answers for each part below.

[8] (a) Find the x and y intercepts of f for $0 \le x \le 2\pi$.

[7] (b) Find all relative maxima and minima of f for $0 \le x \le 2\pi$.

[7] (c) Find all points of inflection of f for $0 \le x \le 2\pi$.

[8] (d) Sketch the graph of f for $0 \le x \le 2\pi$ on the coordinate system below.

9. For the function $f(x) = x^2 - 18 \ln x$, answer the following questions.

[6] (a) Find an equation of the tangent line to the graph of f at the point (1, 1).

[7] (b) Prove that f has at least one root on the closed interval [1, 3].

[7] (c) Prove that f has at most one root on the closed interval [1, 3].

10. Consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

[6] (a) Notice that $f(x)$ is not defined at $x = 2$. Does this create any difficulties in using the precise definition of the limit to prove that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$? Explain why or why not.

[9] (b) Use the precise definition of the limit to prove that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

[15] 11. Let $a < b$ and suppose that f is a function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), with $f(a) = f(b) = 0$. Form a new function $g(x) = e^{-x} f(x)$. Use the function $g(x)$ to show that there is a number c, with $a < c < b$, such that

$$f'(c) = f(c).$$

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MAT 21A - COMMON

Math 21A **Fall 1991** **Final Exam**

I. (28 points) Differentiate each of the following with respect to x. Do NOT simplify your answers.

(a) $y = x^3 + \dfrac{4}{\sqrt[5]{x}} + e^{-x^2/2} + \tan(2\sin 3x)$

(b) $y = \dfrac{7x^2 + 5\ln(x)}{1 + 2\sqrt{x}}$

(c) $y = e^{(4x+1)}\arctan(x^2 + 2)$ [NOTE: $\arctan z = \tan^{-1} z$.]

(d) $y = \ln(\cos(x^2))$

II. The altitude (in miles) of an airplane during a three hour flight is given by

$$s(t) = 3t^2 - t^3,$$

where t is in hours.

(a) (3 points) What is the altitude of the plane when $t = 1$ hour?

(b) (4 points) What is the average rate of change of the plane's altitude between $t = 1$ and $t = 2$ hours?

(c) (5 points) What is the instantaneous rate of change of the plane's altitude when $t = 3$ hours?

(d) (8 points) What is the maximum altitude attained by the plane during the 3 hour flight?

III. (15 points) The curve given by

$$(x-1)^3 + y^3 - 6(x-1)y = 0$$

is called a folium of Descartes. Find the equation of the tangent line to this curve at the point (4,3).

IV. For the function $f(x) = \dfrac{2 + x - x^2}{(x-1)^2}$, $f'(x) = \dfrac{x-5}{(x-1)^3}$, and $f''(x) = \dfrac{-2(x-7)}{(x-1)^4}$.

For each part below, justify your answers. Find

(a) (6 points) the x and y intercepts,

(b) (6 points) all asymptotes,

(c) (6 points) all relative maxima and minima,

(d) (4 points) all points of inflection, and

(e) (8 points) sketch the graph of the curve on the coordinate system below.

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V. **(21 points) For each of the following, either find the desired limit or state that "no limit exists." SHOW ALL WORK NEEDED TO OBTAIN YOUR ANSWERS.**

(a) $\lim_{x \to 4} \frac{64x - x^4}{x^2 - x - 12}$ (b) $\lim_{x \to \infty} \left[x - \sqrt{x^2 - 6x + 1} \right]$

(c) $\lim_{x \to 0^+} x^x$

VI. Let

$$f(x) = \begin{cases} x^3 & \text{, if } x < 0 \\ x^2 & \text{, if } 0 \le x \le 1 \\ x^3 & \text{, if } x > 1 \end{cases}.$$

(a) (10 points) Prove that f(x) is continuous for all real x.

(b) (10 points) Is f(x) differentiable at x = 1? Justify your conclusion, using the <u>definition</u> of the derivative.

VII. (20 points) According to airline regulations, the sum of the dimensions for carry–on luggage cannot exceed 45 inches; i.e., length + width + height $\le$ 45 inches. Determine the dimensions of a carry–on bag that will have a length which is twice its width, have the largest volume possible, and meet the regulations.

VIII. (a) (6 points) State the Intermediate Value Theorem.

(b) (10 points) Use the Intermediate Value Theorem to prove that the polynomial

$$p(x) = 7x^6 + 2x - 5$$

has a real root; i.e., that there exists a real number x_0 such that $p(x_0) = 0$.

IX. (15 points) Using the precise δ–ϵ definition of the limit, prove that

$$\lim_{x \to 2} \left[\frac{3}{7 - x} \right] = \frac{3}{5}.$$

X. (15 points) Consider the function $f(x) = \ln(\sin x)$ on the closed interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. Show that there is at least one number c, with $\frac{\pi}{6} < c < \frac{\pi}{2}$, such that

$$\pi \cot c = 3 \ln 2.$$

[25 pts.] 1. Find $f'(x)$, but do <u>not</u> simplify final answer.

(a) $f(x) = (8x^3 + \tan(x^2))^{151}$

(b) $f(x) = \sec(4x^3 + 2)\sqrt{1 + \cos(x^5)}$.

(c) $f(x) = \dfrac{3 + \tan^{-1} x}{2 + \ln(x^2+1)}$ (Note: $\tan^{-1} x = \text{arc} \tan x$)

(d) $f(x) = (\sin x)^{e^x}$

[15 pts.] 2. Use the definition of the derivative (and not any differentiation formulae) to find $f'(x)$ for

$$f(x) = \frac{2x}{3x + 5}.$$

[15 pts.] 3. Assume that the equation $xy + e^{x^2} \ln y = 5 \sin x$ defines a function $y = f(x)$.

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the line tangent to the graph of $y = f(x)$ at the point $(0,1)$.

[15 pts.]

4. (a) Suppose that $f(x) = \begin{cases} g(x) & \text{if } a \le x < b \\ h(x) & \text{if } b \le x \le c, \end{cases}$

where g is continuous on $[a,b)$ and h is continuous on $[b,c]$. What conditions are needed to ensure that f is continuous on $[a,c]$?

4. (b) Suppose $f(x) = \begin{cases} x + 2c & \text{if } x < -2 \\ 3cx + k & \text{if } -2 \le x \le 1 \\ 3x - 2k & \text{if } 1 < x \end{cases}$

Find the values of the numbers c and k which make f a continuous function, explaining your answer in detail.

[25 pts.] 5. Let $g(x) = \dfrac{x}{1 + x^2}$.

(a) Find the intercepts of the graph of g .

(b) Find the asymptotes.

(c) Find the relative minima.

(d) Find the relative maxima.

(e) Find all inflection points.

(f) Draw the graph of g, showing the information found above.

[15 pts.]

6. The height and diameter of a circular cylinder are equal and both are measured with an error of 2%. Using differentials, find the approximate percent error this may cause in calculating the volume of the cylinder.

[15 pts.]

7. (a) State Rolle's theorem.

(b) Use Rolle's theorem to show that if a car leaves Sacramento from rest, and travels with a velocity which is differentiable with respect to time, and stops in Davis, then its acceleration at some point between Sacramento and Davis was zero.

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[15 pts.] 8. Prove that the equation

$$\frac{x^2 + 1}{x + 3} = \frac{x^4 + 1}{4 - x}$$

has a solution in the interval [0,1].

[20 pts.] 9. Find the following limits.

(a) $\lim_{x \to 0} \left[\frac{3 \sin x}{x^5} - \frac{\cos x + 2}{x^4} \right]$

(b) $\lim_{x \to 0} \left[\frac{1}{x^2} \right]^{\tan x}$

[15 pts.]

10. Two lines tangent to the graph of $g(x) = \frac{x}{\sqrt{x^2 - 16}}$ are parallel to the line $y + 2x + 5 = 0$. Find the coordinates of the points on the graph of g where the lines are tangent to the graph.

[25 pts.]

11. A page of print is to have 54 square inches of printed area, a margin of 2 inches at the bottom, and a margin of 1 inch on the sides and the top. What are the dimensions of the sheet of paper with the smallest area that will accommodate these conditions?

1. Determine the following limits.

6 pts a) $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - x}$

6 pts b) $\lim_{x \to 0} \frac{3x \sin x}{1 - \cos^2 x}$

6 pts c) $\lim_{x \to -\infty} \frac{2x}{\sqrt{x^2 - 1}}$

6 pts d) $\lim_{x \to 0} \left(\frac{1 + 3x}{2}\right)^{1/x}$

2. Consider the function $f(x) = \sqrt[3]{1 + x^5}$.

10 pts a) Verify that the function f is one-to-one.

10 pts b) Find the inverse function for f.

3. Differentiate the following functions. Do NOT simplify your answers.

6 pts a) $f(x) = x^8 \ln (3 + e^x)$

6 pts b) $f(x) = \frac{7x^3 + e^{-x}}{x^2 + \sec 3x}$

6 pts c) $f(x) = (7 + \arctan \sqrt{x}\,)^{10}$

8 pts d) $f(x) = (\sin x)^{\ln x}$

4. a) State the definition of the derivative of a function f at the point x.
5 pts

15 pts b) Use the definition in part a) to compute $f'(x)$ for

$$f(x) = 3 + \frac{1}{x}$$

5. Find the slope of the line tangent to the curve

22 pts $$\ln (y + 1) + (x^2 + y)^3 = 1 + \sin (xy)$$

at the point (1, 0).

6. Show that there are exactly two solutions to the equation

18 pts $$e^x - x = 2.$$

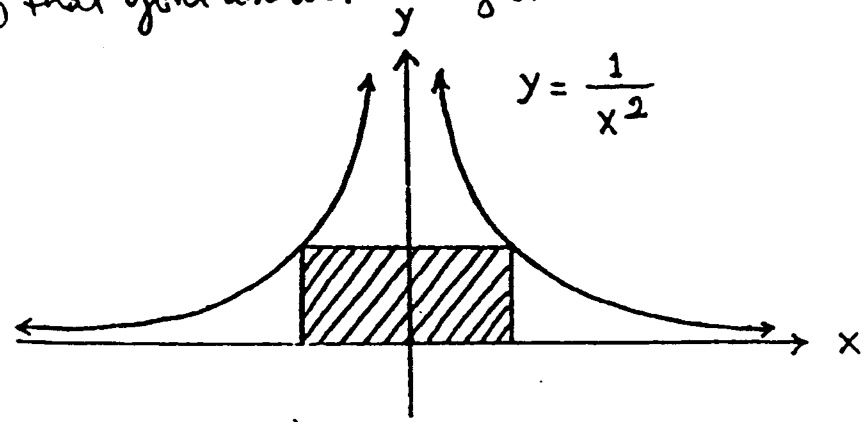
#7 ON NEXT PAGE ...

8. The following function f is continuous at $x = 0$ for all values of
20 pts k. Determine a value for k so that f is <u>differentiable</u> at $x = 0$.

$$f(x) = \begin{cases} 4 + kx - x^2, & x < 0 \\ 4 + (7 + 3k)\,x, & x \geq 0 \end{cases}.$$

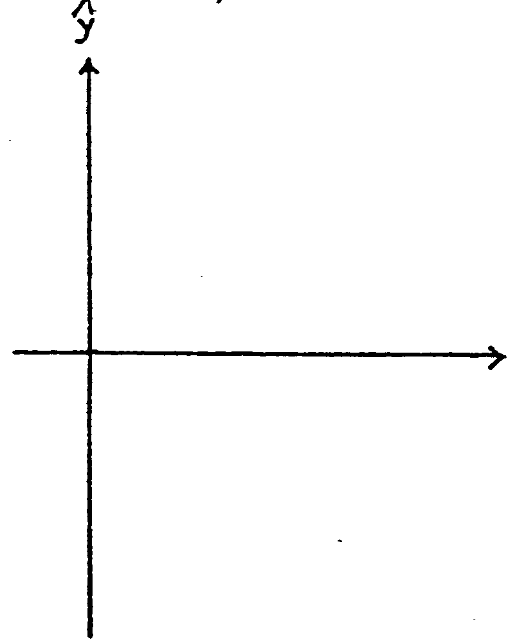
7. 22 pts Consider all rectangles inscribed in such a way that their bases lie on the x-axis with the top corners on the graph of $y = \frac{1}{x^2}$. Find the length and width of the inscribed rectangle of minimum perimeter.

Verify that your answer is a global minimum



9. Let $f(x) = 7\,x\,e^{-2x}$ for $x \geq 0$.

15 pts a) Sketch the graph of f for $x \geq 0$. Label all relative extrema, inflection points, and asymptotes.



2 pts b) On the same graph, sketch the line $y = x$.

5 pts c) State the hypotheses of Rolle's Theorem.

6 pts d) Find an interval on which the function $y = 7\,x\,e^{-2x} - x$ for $x \geq 0$ satisfies the hypotheses of Rolle's Theorem.

Mathematics 21A Final Exam FALL 1988

1. Determine the following limits.

9 pts. a) $\lim_{x\to 0} \frac{\tan^{-1} 2x}{\tan^{-1} 5x}$

7 pts. b) $\lim_{x\to 0} \frac{x}{e^x + 1}$

9 pts. c) $\lim_{x\to\infty} (1 + \frac{5}{x})^{\frac{x}{3}}$

2. Find the derivatives of the following functions. Do NOT simplify your answers.

8 pts. a) $f(x) = \frac{(\cos 2x)^5}{\sqrt{1 - x^2}}$

9 pts. b) $\ln(\sec^2(\frac{1}{x}))$

8 pts. c) $(3x - 1)^{2x+1}$

3. Let $f(x) = x^2 + x$.

12 pts. a) Use the definition of the derivative to compute $\frac{df}{dx}$.

7 pts. b) Let $f(x) = x^2 + x$ be the mass (in grams) in the interval $[0,x]$ of a thin rod (the length is measured in centimeters). What is the average density over the interval $[1, 3]$?

6 pts. c) What is the density at $x = 1$ for the rod in part (b)?

20 pts. 4. Use the differential to find an approximation of $e^{\sin 2x}$ when x is near 0.

5. Let $f(x) = \frac{1}{x^3 - 3x}$.

5 pts. a) Determine the vertical asymptotes of f, if any. (The result of your work in parts a)–d) is to be later incorporated in the graph in part (e).)

4 pts. b) Determine the horizontal asymptotes of f, if any.

6 pts. c) Determine the critical points of f, if any.

7 pts. d) Determine where f is increasing and decreasing, and find all the local maximum and local minimum points of f.

→ MORE FUN ON NEXT PAGE

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5, CONT'D 8 pts. e) Sketch the graph of f, indicating all asymptotes and maximum and minimum points.

6.

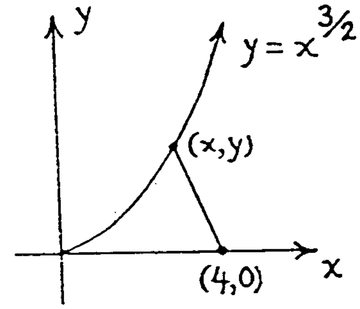
8 pts. a) State the hypotheses and the conclusion of the Mean Value Theorem.

12 pts. b) Let f be a function which is differentiable for all x and such that $f(1) = 2$. Also $f'(x) \leq 3$ for all x. Show that $f(4) \leq 11$.

7.

20 pts a) Find the coordinates of the point on the curve $y = x^{3/2}$ which is closest to the point $(4,0)$.

10 pts. b) Use an appropriate test to verify that the answer obtained in part (a) results in a local minimum.



8.

18 pts. a) Let $f(x) = \begin{cases} \frac{1}{x} & \text{if } x < -1 \\ \frac{|x|}{x} & \text{if } -1 < x < 0 \\ e^{-\sin x} & \text{if } 0 \leq x < \frac{\pi}{2} \\ \frac{1}{e} & \text{if } \frac{\pi}{2} \leq x \end{cases}$

Using the definition of continuity, determine whether or not f is continuous at the following three points (show how you decide):

7 pts. b) Is the following statement true or false? (justify your answer)

If g is a function defined on $[0,1]$, and for every $0 < c < 1$ $g(c)$ is between $g(0)$ and $g(1)$, then g is continuous on $[0,1]$.

MATH 21A - ALL COMMON FINAL

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1. (6 PTS) A. FIND THE DOMAIN & RANGE OF $g(x) = \dfrac{1}{-2 + \sqrt{x-3}}$.

(10 PTS) B. CONSIDER $w(x) = \dfrac{x^2 - 4}{x - 2}$.

a. FIND THE DOMAIN OF w.

b. IS W CONTINUOUS AT $x = 1$? BRIEFLY JUSTIFY YOUR POSITION.

c. IS THERE A VALUE, b, SUCH THAT $W(x)$ DEFINED BY

$$W(x) = \begin{cases} w(x) & \text{FOR } x \neq 2 \\ b & \text{FOR } x = 2 \end{cases}$$

IS CONTINUOUS FOR ALL REAL x? JUSTIFY YOUR POSITION.

II. (21 points) For each of the following, either find the desired limit or state that "no limit exists." JUSTIFY YOUR ANSWERS.

(a) $\lim\limits_{x \to 2} \dfrac{x^2 + 6x}{x - 2}$ (b) $\lim\limits_{x \to 0} \dfrac{\sin 2x - \tan 3x}{4x\cos x}$ (c) $\lim\limits_{x \to \infty} \dfrac{\sqrt{3x^2 + 2x} - \sqrt{x^2 + 4}}{x}$

III. (a) (5 points) State the definition of the derivative of a function $y = f(x)$ at a point $x = a$.

(b) (10 points) Use the definition of the derivative to find the derivative of
$$f(x) = x^3 + 5x.$$

IV. (40 points) Differentiate each of the following with respect to x. Do NOT simplify your answers.

(a) $y = \sqrt{(\frac{1}{2})(x^2 + 4)^3}$ (b) $y = \dfrac{xe^{(x^2+4)}}{2 + x^2}$

(c) $y = \sin[\ln(6x^2 - 7)]$ (d) $y = (\tan x)^{(4/x^2)}$

V. For the function $f(x) = \dfrac{x^2}{2} + \dfrac{8}{x}$, find

(a) (4 points) the x and y intercepts, (b) (4 points) the asymptotes,

(c) (4 points) the relative maxima and minima.

(d) (4 points) the intervals where the function is increasing and where it is decreasing.

→ MORE TO COME . . .

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(e) (4 points) the points of inflection,

(f) (8 points) sketch the graph of the curve on the coordinate system provided below.

VI. (15 points) Given that y is a function of x that satisfies

$$x^3 + x^2y - xy^3 + 5 = 0,$$

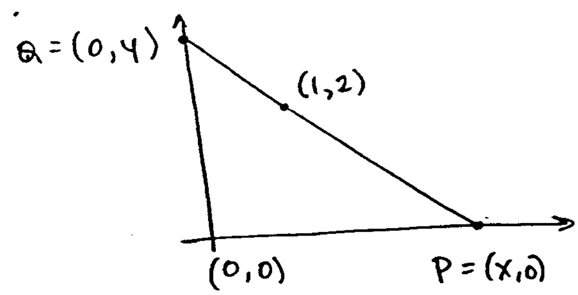
find y′ and y′′ at the point (1,2).

VII. (10 points) While standing on top of a tower that is 96 feet above the ground, a boy threw a rock upwards. The rock hit the ground 6 seconds later. At what speed was the rock traveling when it left the boy's hand? [Recall that the acceleration due to gravity is -32 ft/sec^2.]

VIII. (15 points) Apply the Mean Value Theorem to the function $f(x) = \ell n x$ in the interval $[1,b]$ to prove that if $b > 1$, then

$$1 - \frac{1}{b} < \ell n(b) < b - 1.$$

IX. (20 PTS) GIVEN A LINE PASSING THROUGH THE POINT (1,2) AND INTERSECTING THE X-AXIS AND Y-AXIS AT THE POINTS $P = (x,0)$ AND $Q = (0,y)$, RESPECTIVELY, LET ΔOPQ DENOTE THE TRIANGLE WITH VERTICES P, Q, AND O = (0,0) LYING IN THE FIRST QUADRANT. FIND THE MINIMUM AREA AMONG AREAS OF ALL SUCH TRIANGLES AND THE EQUATION OF THE LINE THAT GIVES THE MINIMUM AREA.



X. Let $f(x) = x^5 + x + 1$.

(a) (11 points) Show that the equation $f(x) = 0$ has exactly one root in the interval $[-1,1]$.

(b) 6 pts. USING $x_1 = 0$ AS A FIRST ESTIMATE, APPLY NEWTON'S METHOD TO FIND A SECOND ESTIMATE x_2 FOR THE ROOT.