HW #4

Section 2.8

$\boxed{83:1}$ i.) $f(\frac{1}{2}) = 1$,
ii) $\lim_{x \to \frac{1}{2}} f(x) = 1$,
and iii) $\lim_{x \to \frac{1}{2}} f(x) = f(\frac{1}{2})$

so f is continuous at $x = \frac{1}{2}$.

$\boxed{83:2}$ i.) $f(\frac{1}{2}) = \frac{1}{2}$,
ii) $\lim_{x \to \frac{1}{2}} f(x) = 1$,
but iii) $\lim_{x \to \frac{1}{2}} f(x) \neq f(\frac{1}{2})$

so f is not continuous at $x = \frac{1}{2}$.

$\boxed{83:3}$ i.) $f(\frac{1}{2}) = \frac{1}{2}$,
ii) $\lim_{x \to \frac{1}{2}} f(x)$ does not exist,
and iii) $\lim_{x \to \frac{1}{2}} f(x) \neq f(\frac{1}{2})$

so f is not continuous at $x = \frac{1}{2}$.

$\boxed{83:4}$ i.) $f(1) = \frac{1}{2}$,
ii) $\lim_{x \to 1^-} f(x) = +\infty$
and iii) $\lim_{x \to 1^-} f(x) \neq f(1)$

so f is not continuous at $x = 1$.

$\boxed{83:5}$ i.) $f(0) = 0$,
ii) $\lim_{x \to 0} f(x)$ does not exist,
and iii) $\lim_{x \to 0} f(x) \neq f(0)$

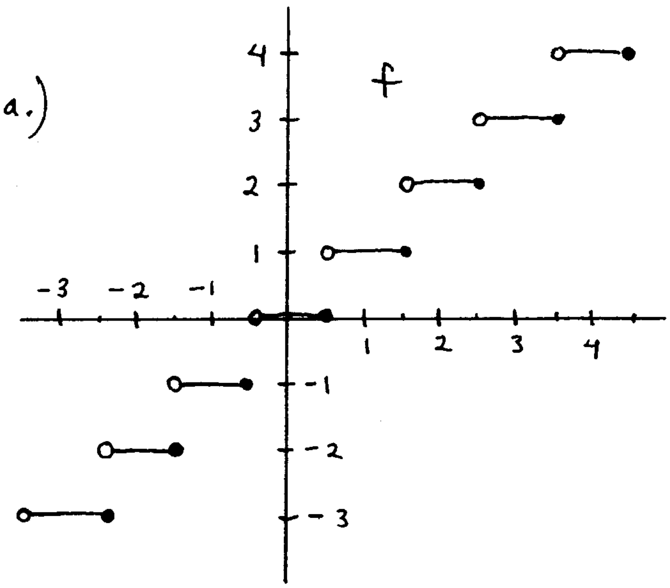
so f is not continuous at $x = 0$.

$\boxed{83:6}$ i.) $f(0) = 0$,
ii) $\lim_{x \to 0} f(x) = 0$,

and iii) $\lim_{x \to 0} f(x) = f(0)$ so f is continuous at $x = 0$.

$\boxed{83:12}$

a.)



b.) $\lim_{x \to 3.5^-} f(x) = 3$

c.) $\lim_{x \to 3.5^+} f(x) = 4$

d.) $\lim_{x \to 3.5} f(x)$ does not exist

e.) f is not continuous at $x = 3.5$

g.) ~~f.)~~ f is continuous everywhere except $x = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$

$\boxed{83:13}$ $f(x) = \dfrac{1 - \cos x}{x}$ for $x \neq 0$; since

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos x}{x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = (1) \cdot \left(\frac{0}{2}\right) = 0 ,$$

f will be continuous at $x = 0$ if we define $f(0) = 0$. Note that f is already continuous for all $x \neq 0$ since $1 - \cos x$ and x are continuous for all x-values (and $x \neq 0$)

$\boxed{83:14}$ $f(x) = \dfrac{x^3-1}{x-1}$ for $x \neq 1$; since $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^3-1}{x-1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 3$, f will be continuous at $x=1$ if we define $f(1)=3$. Note that f is already continuous for all $x \neq 1$.

$\boxed{83:38}$ $f(x) = 2^{\frac{1}{x}}$ for $x \neq 0$

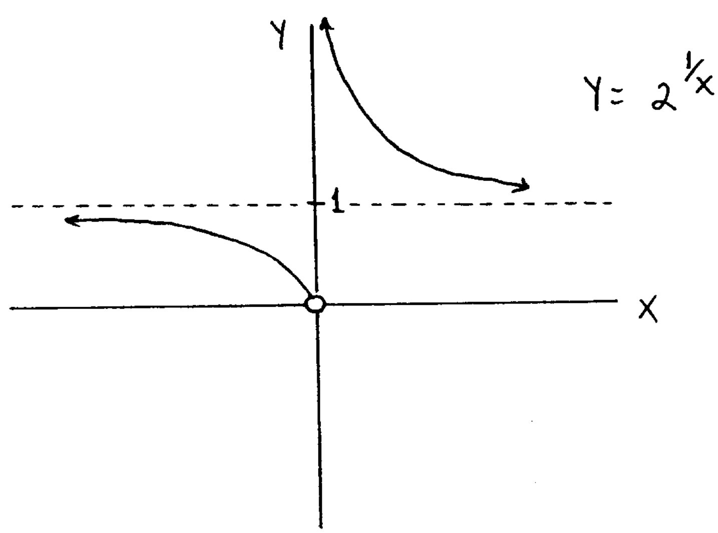
a.) $\lim\limits_{x\to +\infty} 2^{1/x} = 2^0 = 1$

b.) $\lim\limits_{x\to -\infty} 2^{1/x} = 2^0 = 1$

c.) $\lim\limits_{x\to 0^+} 2^{\frac{1}{x}} = 2^{\text{"}\frac{1}{0^+}\text{"}} = \text{"}2^{+\infty}\text{"} = +\infty$ (does not exist)

d.) $\lim\limits_{x\to 0^-} 2^{\frac{1}{x}} = 2^{\text{"}\frac{1}{0^-}\text{"}} = \text{"}2^{-\infty}\text{"} = \text{"}\frac{1}{2^{+\infty}}\text{"} = \text{"}\frac{1}{\infty}\text{"} = 0$

e.)



$\boxed{83:48}$ assume $f(x+y) = f(x) + f(y)$ for all numbers x and y; assume $f(1) = c$.

a.) $f(2) = f(1+1) = f(1) + f(1) = c + c = 2c$

b.) $f(0) = f(0+0) = f(0) + f(0) = 2f(0) \longrightarrow$

$f(0) = 2f(0) \to f(0) = 0$.

c.) $f(0) = f(1+(-1)) = f(1) + f(-1) \to 0 = c + f(-1) \to$
$f(-1) = -c$.

d.) $f(n) = f(\underbrace{1+1+\cdots+1}_{n \text{ times}}) = \underbrace{f(1)+f(1)+\cdots+f(1)}_{n \text{ times}}$
$= n \cdot f(1) = n \cdot c$ for n a positive integer.

e.) If n is a negative integer then
$f(0) = f(n + (-n)) = f(n) + f(-n) \to$
a positive integer
$0 = f(n) + -n \cdot c \to f(n) = nc$.

f.) $f(1) = f(\frac{1}{2}+\frac{1}{2}) = f(\frac{1}{2}) + f(\frac{1}{2}) = 2f(\frac{1}{2}) \to$
$c = 2f(\frac{1}{2}) \to f(\frac{1}{2}) = c/2$.

g.) $f(1) = f(\underbrace{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}}_{n \text{ times}}) = \underbrace{f(\frac{1}{n})+f(\frac{1}{n})+\cdots+f(\frac{1}{n})}_{n \text{ times}} \to$
$c = n \cdot f(\frac{1}{n}) \to f(\frac{1}{n}) = c/n$.

h.) $f(\frac{m}{n}) = f(m \cdot \frac{1}{n}) = f(\underbrace{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}}_{m \text{ times}}) \to$
$f(\frac{m}{n}) = \underbrace{f(\frac{1}{n})+f(\frac{1}{n})+\cdots+f(\frac{1}{n})}_{m \text{ times}} = m f(\frac{1}{n}) = m \cdot c/n$.

i.) If x is irrational then $\lim_{n\to\infty} x_n = x$ where x_n is a sequence of rational numbers so
$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} c \cdot x_n = cx$.

$\boxed{83:49}$ Sometimes "0^0" is 1 and sometimes it isn't:

a.)

x	$\sin x$	$\sqrt{x}$	$(\sin x)^{\sqrt{x}}$
1	0.8414709	1	0.8414709
0.1	0.0998334	0.3162277	0.482551
0.01	0.0099998	0.1	0.6309562
0.001	0.0009999	0.0316227	0.8037697
0.0001	0.0001	0.01	0.9120108
0.0000001	0.0000001	0.0003162	0.9949159

Here, $\lim_{x\to 0} (\sin x)^{\sqrt{x}} = \text{"}0^0\text{"} = 1$

b.)

x	$2^{-1/x}$	$-x$	$(2^{-1/x})^{-x}$
1	0.5	-1	2
0.1	0.0009765	-0.1	2
0.01	7.8×10^{-31}	-0.01	2
0.001	tiny	-0.001	2

Here, $\lim_{x\to 0} \left(2^{-1/x}\right)^{-x} = \text{"}0^0\text{"} = 2$!

Math 21A
Kouba
Worksheet 1

1.) Use limits and algebra to determine the value of constants A and B so that each of the following functions is continuous for all values of x.

a.) $f(x) = \begin{cases} \dfrac{x^2 - 7x + 6}{x - 6}, & \text{if } x \neq 6 \\ A, & \text{if } x = 6. \end{cases}$

b.) $f(x) = \begin{cases} A^2 x - A, & \text{if } x \geq 1 \\ 2, & \text{if } x < 1. \end{cases}$

c.) $f(x) = \begin{cases} \dfrac{A + x}{A + 1}, & \text{if } x < 0 \\ Ax^3 + 3, & \text{if } x \geq 0. \end{cases}$

d.) $f(x) = \begin{cases} 3, & \text{if } x \leq 1 \\ Ax^2 + B, & \text{if } 1 < x \leq 2 \\ 5, & \text{if } x > 2. \end{cases}$

e.) $f(x) = \begin{cases} Ax - B, & \text{if } x \leq -1 \\ 2x + 3A + B, & \text{if } -1 < x \leq 1 \\ 4, & \text{if } x > 1. \end{cases}$

2.) For what x-values are the following functions continuous ? Briefly explain why using shortcuts and rules from class. Sketch the graph of each using a graphing calculator.

a.) $g(x) = \dfrac{x + 1}{x^2 - 4}$

b.) $h(x) = \dfrac{100}{4 - \sqrt{x^2 - 9}}$

c.) $h(x) = \sin^3(\ln(3x - 5))$

d.) $g(x) = \begin{cases} \dfrac{x^2 - 3x - 4}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4. \end{cases}$

e.) $f(x) = \begin{cases} \dfrac{x^3 + 1}{x^2 - 1}, & \text{if } x \neq 1, -1 \\ -3/2, & \text{if } x = -1 \\ 3, & \text{if } x = 1\ . \end{cases}$

Worksheet 1 Solutions

1.) a.) Since $\lim_{x\to 6} f(x) = \lim_{x\to 6} \frac{x^2-7x+6}{x-6}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 6} \frac{\cancel{(x-6)}(x-1)}{\cancel{(x-6)}} = 5$, choosing $\boxed{a=5}$

makes f continuous at $x=6$ (it's already continuous for $x \neq 6$.)

b.) f is continuous for $x<1$ and for $x>1$. We must make f continuous at $x=1$:

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (a^2x-a) = a^2-a$ and

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2) = 2$, thus $a^2-a=2 \rightarrow$

$a^2-a-2=0 \rightarrow (a-2)(a+1)=0 \rightarrow \boxed{a=2}$ or $\boxed{a=-1}$

c.) f is continuous for $x<0$ (so long as $a \neq -1$) and for $x>0$. We must make f continuous at $x=0$:

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (ax^3+3) = 3$ and

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{a+x}{a+1} = \frac{a}{a+1}$, thus $\frac{a}{a+1} = 3 \rightarrow$

$a = 3a+3 \rightarrow -3 = 2a \rightarrow a = \frac{-3}{2}$.

d.) f is continuous for $x<1$, for $1<x<2$, and for $x>2$. We must make f continuous at $x=1$ and at $x=2$:

<u>at $x=1$</u> : $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (ax^2+b) = a+b$ and

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (3) = 3$, thus $\boxed{a+b=3}$;

<u>at $x=2$</u> : $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (5) = 5$ and

$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (ax^2+b) = 4a+b$, so $\boxed{4a+b=5}$;

thus $\left.\begin{array}{l} a+b=3 \\ 4a+b=5 \end{array}\right\}$ $b = 3-a$ $\rightarrow 4a+(3-a)=5 \rightarrow$

$3a=2 \rightarrow$ $a = 2/3$ and $b = \frac{7}{3}$.

e.) f is continuous for $x<-1$, for $-1<x<1$, and for $x>1$. We must make f continuous at $x=-1$ and $x=1$:

<u>at $x=-1$</u> : $\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} (2x+3a+b) = 3a+b-2$ and

$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} (ax-b) = -a-b$, so $\boxed{3a+b-2=-a-b}$;

<u>at $x=1$</u> : $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (4) = 4$ and

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (2x+3a+b) = 2+3a+b$, so $\boxed{2+3a+b=4}$;

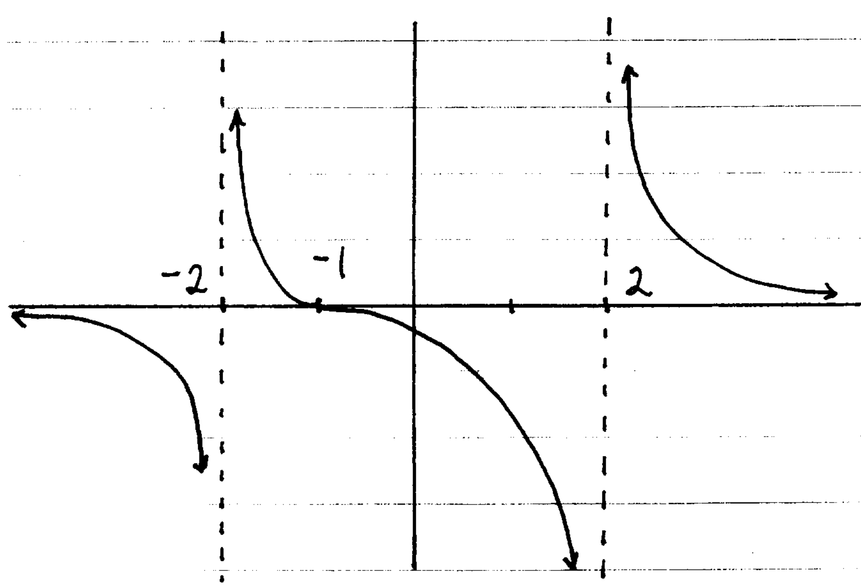
thus, $\left.\begin{array}{l}3a+b-2=-a-b \\ 2+3a+b=4\end{array}\right\}$ $\left.\begin{array}{l}4a+2b=2 \\ 3a+b=2\end{array}\right\}$ $b=2-3a$

$\rightarrow 4a+2(2-3a)=2 \rightarrow 4a+4-6a=2 \rightarrow$

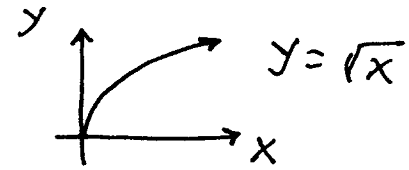
$2=2a \rightarrow$ $a=1$ and $b=-1$

2.) a.) $y=x+1$ and $y=x^2-4$ are continuous for all values of x (since they are polynomials), so $g(x)=\dfrac{x+1}{x^2-4}$ is

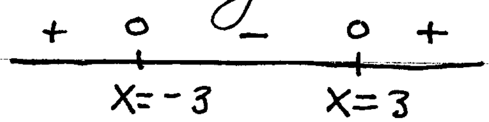
continuous for all values of x (quotient of continuous functions) except where $x^2-4=(x-2)(x+2)=0$, i.e., except for $x=2$ and $x=-2$.



b.) $y=x^2-9$ and $y=100$ are continuous for all values of x (since they are polynomials); $y=\sqrt{x}$ is a well



known continuous function for $x \geq 0$; let $f(x) = \sqrt{x}$ and $g(x) = x^2 - 9$, then $\sqrt{x^2-9} = f(g(x))$ is continuous (composition of continuous functions) so long as $x^2 - 9 \geq 0$, i.e., $(x-3)(x+3) \geq 0$,



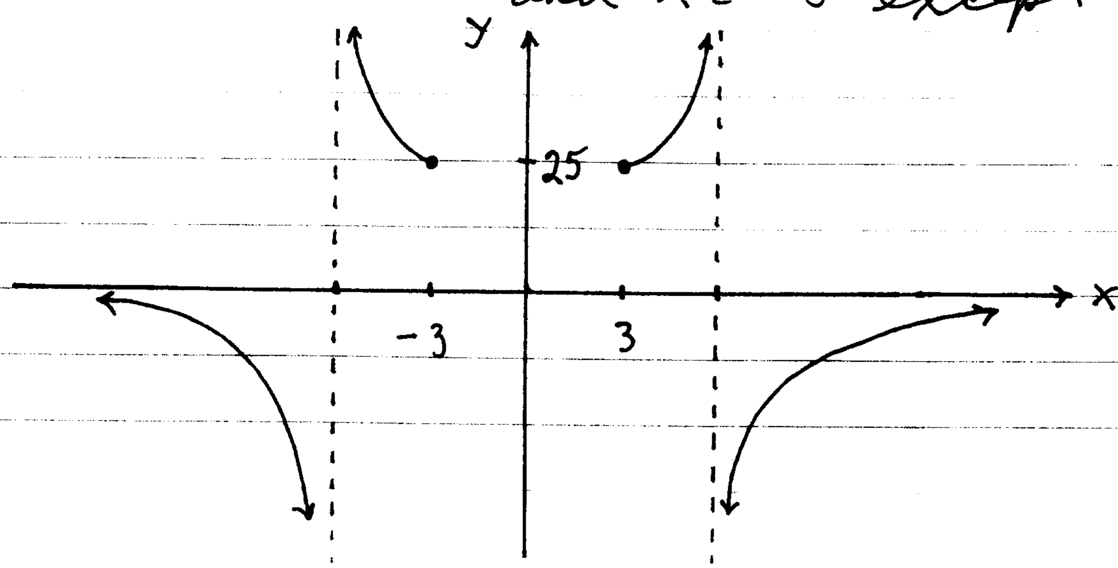
i.e., for $x \geq 3$ and $x \leq -3$;

$y = 4$ is continuous for all values of x, so that $y = 4 - \sqrt{x^2-9}$ is continuous (difference of continuous functions) for $x \geq 3$ and $x \leq -3$; finally,

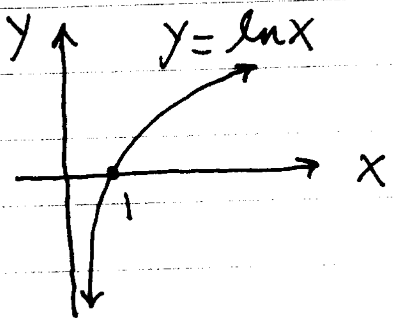
$h(x) = \dfrac{100}{4-\sqrt{x^2-9}}$ is continuous (quotient of continuous functions) for $x \geq 3$ and $x \leq -3$ so long as $4 - \sqrt{x^2-9} \neq 0$;

$4 - \sqrt{x^2-9} = 0 \Rightarrow 4 = \sqrt{x^2-9} \Rightarrow 16 = x^2 - 9 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$; thus,

$h(x) = \dfrac{100}{4-\sqrt{x^2-9}}$ is continuous for $x \geq 3$ and $x \leq -3$ except $x = \pm 5$.



c.) $y = 3x-5$ and $y = x^3$ are continuous for all values of x (since they are polynomials), and $y = \sin x$ is a well known function continuous for all values of x; $y = \ln x$ is a well



known function continuous for $x > 0$; let $f(x) = \ln x$ and $g(x) = 3x-5$, then $\ln(3x-5) = f(g(x))$ is continuous (composition of continuous functions) so long as $3x-5 > 0$, i.e., for $x > 5/3$; let $k(x) = x^3$ and $l(x) = \sin x$ then $h(x) = \sin^3(\ln(3x-5)) = k(l(f(g(x))))$ is continuous (composition of continuous functions) for $x > 5/3$.

For graph of function try the following ranges for x:

1. $5/3 < x \le 1000$
2. $5/3 < x \le 100$
3. $5/3 < x \le 10$
4. $5/3 < x \le 2$
5. $5/3 < x \le 1.75$
6. $5/3 < x \le 1.68$
7. $5/3 < x \le 1.668$
8. $5/3 < x \le 1.6668$

d.) $g(x) = \begin{cases} \dfrac{x^2-3x-4}{x-4} & , \text{if } x \neq 4 \\ 5 & , \text{if } x = 4 \end{cases}$

$= \begin{cases} \dfrac{(x-4)(x+1)}{x-4} & , \text{if } x \neq 4 \\ 5 & , \text{if } x = 4 \end{cases}$

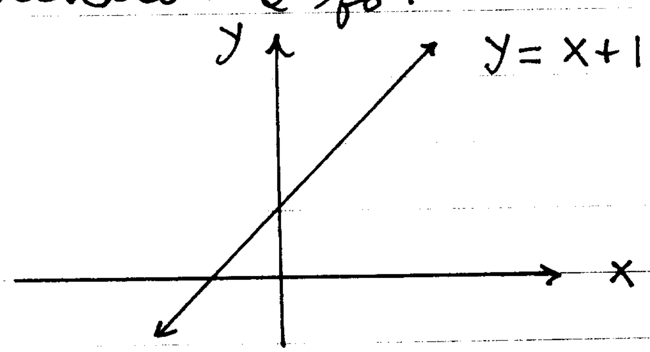
$= \begin{cases} x+1 & , \text{if } x \neq 4 \\ 5 & , \text{if } x = 4 \end{cases}$;

i.) $g(4) = 5$

ii.) $\lim\limits_{x \to 4} g(x) = \lim\limits_{x \to 4} (x+1) = 4+1 = 5$

iii) $\lim\limits_{x \to 4} g(x) = g(4)$;

thus g is continuous at $x = 4$;
since $y = x+1$ is continuous for
$x \neq 4$ (since it is a polynomial),
g is continuous for all values of x.



e.) $f(x) = \begin{cases} \dfrac{x^3+1}{x^2-1} & , \text{if } x \neq 1, -1 \\ -3/2 & , \text{if } x = -1 \\ 3 & , \text{if } x = 1 \end{cases}$

$y = x^3+1$ and $y = x^2-1$ are continuous for all values of x (since they are polynomials), so $y = \frac{x^3+1}{x^2-1}$ is continuous for all values of x except where $x^2-1=0$, i.e., except for $x = \pm 1$;

<u>check $x=1$</u>: i.) $f(1) = 3$, ii.) $\lim_{x\to 1} f(x)$

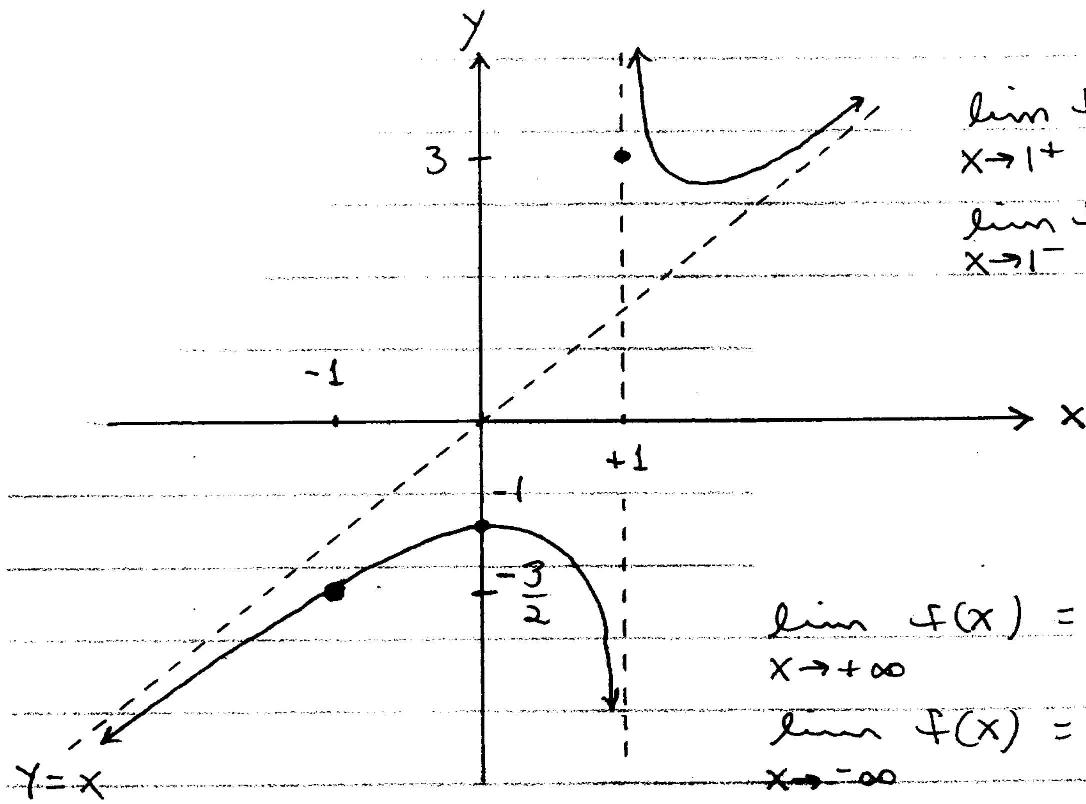
$$= \lim_{x\to 1} \frac{x^3+1}{x^2-1} = \text{"}\frac{2}{0^\pm}\text{"} = \pm\infty$$ so $\lim_{x\to 1} f(x)$

does NOT exist and f is <u>NOT cont. at</u> <u>$x=1$</u>;

<u>check $x=-1$</u>: i.) $f(-1) = \frac{-3}{2}$, ii.) $\lim_{x\to -1} f(x)$

$$= \lim_{x\to -1} \frac{x^3+1}{x^2-1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to -1} \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}(x-1)} = \frac{3}{-2} = \frac{-3}{2}$$

and iii.) $f(-1) = \lim_{x\to -1} f(x)$ so that f is continuous at $x=-1$; thus, f is continuous for all x-values except $x=1$.



$$\lim_{x\to 1^+} f(x) = \text{"}\frac{2}{0^+}\text{"} = +\infty,$$

$$\lim_{x\to 1^-} f(x) = \text{"}\frac{2}{0^-}\text{"} = -\infty,$$

$$\lim_{x\to +\infty} f(x) = +\infty,$$

$$\lim_{x\to -\infty} f(x) = -\infty$$