Math 21A
Kouba
Exam 1 Solutions

1.) volume $V = \frac{4}{3}\pi r^3$, surface area $S = 4\pi r^2$ so

$r^2 = \frac{S}{4\pi} \rightarrow r = \left(\frac{S}{4\pi}\right)^{1/2}$ then

$$V = \frac{4}{3}\pi\left(\frac{S}{4\pi}\right)^{\frac{1}{2}\cdot 3} \text{ or } V = \frac{S^{3/2}}{3}\cdot\frac{4\pi}{(4\pi)^{3/2}} = \frac{S^{3/2}}{3\sqrt{4\pi}}$$

or $V = \frac{1}{6\sqrt{\pi}} S^{3/2}$

2.) $x^2 - 1 \geq 0$ so $x \geq 1$, $x \leq -1$ and

$2 - \sqrt{x^2-1} \neq 0 \rightarrow 2 \neq \sqrt{x^2-1} \rightarrow 4 \neq x^2 - 1 \rightarrow$

$x^2 \neq 5 \rightarrow x \neq \pm\sqrt{5}$ so

Domain: $x \leq -1$, $x \geq 1$ but $x \neq \pm\sqrt{5}$

3.) $f(x) = \frac{x}{x+1}$ then $f(g(x)-1) = x \rightarrow$

$$\frac{(g(x)-1)}{(g(x)-1)+1} = x \rightarrow \frac{g(x)-1}{g(x)} = x \rightarrow g(x) - 1 = x\,g(x) \rightarrow$$

$$g(x) - x\,g(x) = 1 \rightarrow (1-x)\,g(x) = 1 \rightarrow$$

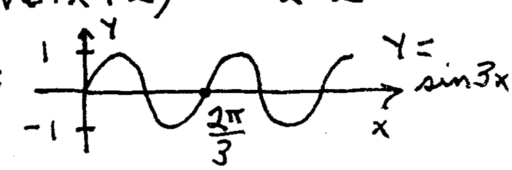
$$g(x) = \frac{1}{1-x}$$

4.) a.) $\lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}(x-2)} \overset{"\frac{0}{0}"}{=} \frac{4}{1} = 4$

b.) $\lim_{x\to -2} \frac{\sqrt{6+x}-2}{x+2}\cdot\frac{\sqrt{6+x}+2}{\sqrt{6+x}+2}$

$$\overset{"\frac{0}{0}"}{=} \lim_{x\to -2} \frac{(6+x)-4}{(x+2)(\sqrt{6+x}+2)} = \lim_{x\to -2}\frac{\cancel{x+2}}{\cancel{(x+2)}(\sqrt{6+x}+2)} = \frac{1}{2+2} = \frac{1}{4}$$

c.) $\lim_{x\to\infty} \sin 3x$ DNE since:



d.) $\lim_{x\to 0} \dfrac{2x-\tan x}{x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0}\left(\dfrac{2\cancel{x}}{\cancel{x}} - \dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}\right)$

$= 2-(1)\cdot\dfrac{1}{\cos 0} = 2-1 = 1$

e.) $\lim_{x\to 1^+} \dfrac{x^3-2x}{x-1} = \text{"}\dfrac{1-2}{0^+}\text{"} = \text{"}\dfrac{-1}{0^+}\text{"} = -\infty$ (DNE)

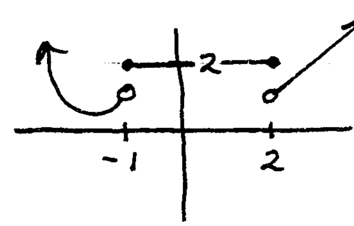
f.) $-1 \le \cos x \le +1 \rightarrow 1 \le 2+\cos x \le 3 \rightarrow$

$\dfrac{1}{x^2} \le \dfrac{2+\cos x}{x^2} \le \dfrac{3}{x^2} \rightarrow \lim_{x\to\infty}\dfrac{1}{x^2} \le \lim_{x\to\infty}\dfrac{2+\cos x}{x^2} \le \lim_{x\to\infty}\dfrac{3}{x^2} \rightarrow$

$0 \le \lim_{x\to\infty}\dfrac{2+\cos x}{x^2} \le 0$ so by Squeeze Principle

$\lim_{x\to\infty}\dfrac{2+\cos x}{x^2} = 0$

FAKE GRAPH



5.) $f(x) = \begin{cases} (Ax)^2+Bx, & x<-1 \\ 2, & -1\le x\le 2 \\ Ax-B+3, & x>2 \end{cases}$

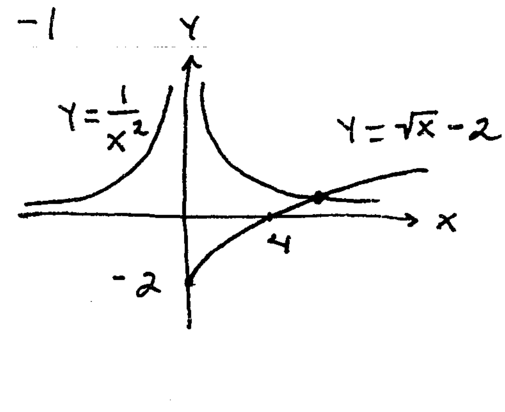
$\left.\begin{array}{l} \lim_{x\to 2^+}(Ax-B+3) = 2 \\ \lim_{x\to -1^-}(Ax)^2+Bx = 2 \end{array}\right\}$ $2A-B+3=2 \rightarrow B=2A+1$

$A^2-B=2 \leftarrow$ $\rightarrow$

$A^2-(2A+1)=2 \rightarrow$

$A^2-2A-3=0 \rightarrow (A-3)(A+1)=0 \rightarrow$

$A=3,\ B=7$ or $A=-1,\ B=-1$



6.) It appears there's a solution larger than 4.

then $\dfrac{1}{x^2} = -2+\sqrt{x} \rightarrow$

$f(x) = \dfrac{1}{x^2}+2-\sqrt{x} = 0 = m$,

$f(4) = \frac{1}{16} + 2 - 2 = \frac{1}{16} > 0$ and

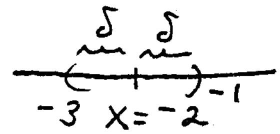
$f(9) = \frac{1}{81} + 2 - 3 = \frac{-80}{81} < 0$ and $m = 0$ is between $f(4)$ and $f(9)$. Since f is a continuous function (~~it~~ it is sum and difference of continuous functions) on the closed interval $[4, 9]$ and $m = 0$ is between $f(4)$ and $f(9)$, it follows from IMVT that there is at least one number c, $4 \le c \le 9$, so that $f(c) = m$, i.e., $\frac{1}{c^2} + 2 - \sqrt{c} = 0$, i.e., $\frac{1}{c^2} = -2 + \sqrt{c}$ so original equation is solvable.

7.) $\lim_{x \to -2} (x^2 - x) = 6$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x+2| < \delta$ then $|f(x) - 6| < \varepsilon$. Begin with $|f(x) - 6| < \varepsilon$ and "solve for" $|x+2|$.

Then $|f(x) - 6| < \varepsilon$ iff $|(x^2 - x) - 6| < \varepsilon$

iff $|(x-3)(x+2)| < \varepsilon$

iff $|x-3||x+2| < \varepsilon$. Eliminate $|x-3|$.

Assume $\delta \le 1$ so that  and

$4 < |x-3| < 6$.

Then $|x-3||x+2| < 6|x+2| < \varepsilon$

iff $6|x+2| < \varepsilon$ iff $|x+2| < \varepsilon/6$.

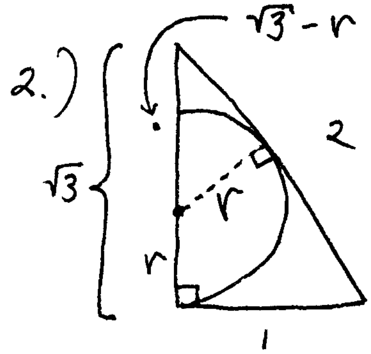
Choose $\delta = \min\{1, \varepsilon/6\}$. Thus, if $0 < |x+2| < \delta$, then $|f(x) - 6| < \varepsilon$. This completes the proof.

Extra Credit:

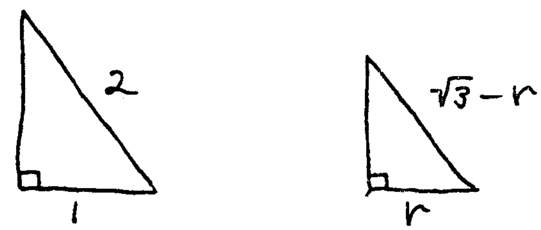
1.) $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 2x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \dfrac{5}{2}\cdot \dfrac{\sin 5x}{5x}\cdot \dfrac{2x}{\sin 2x}$

$= \left(\frac{5}{2}\right)(1)(1) = \frac{5}{2}$

2.)



By similar triangles:



$\frac{2}{1} = \frac{\sqrt{3}-r}{r} \rightarrow 2r = \sqrt{3}-r \rightarrow 3r = \sqrt{3} \rightarrow$

$r = \frac{1}{\sqrt{3}}$