1.) (10 pts. each) Diffferentiate each of the following. DO NOT SIMPLIFY answers.

a.) $y = e^{x^2}\cos^3(5x)$

$$Y' = e^{x^2}\cdot 3\cos^2(5x)\cdot -\sin(5x)\cdot 5$$
$$+ e^{x^2}\cdot 2x\cdot \cos^3(5x)$$

b.) $y = \arcsin(2^x + \log x)$

$$Y' = \frac{1}{\sqrt{1-(2^x+\log x)^2}}\cdot \left\{2^x\cdot \ln 2 + \frac{1}{x}\cdot \log_{10} e\right\}$$

c.) $y = x^{\ln x}$ $\rightarrow \ln Y = \ln x\cdot \ln x = (\ln x)^2 \rightarrow$

$$\frac{1}{Y}Y' = 2(\ln x)\cdot \frac{1}{x} \rightarrow Y' = x^{\ln x}\cdot \frac{2\ln x}{x}$$

2.) (10 pts.) You deposit \$2000 in a retirement account earning 12% annual interest compounded monthly. In how many years will the account grow to \$10,000 ?

$$A = P\left(1+\frac{r}{n}\right)^{nt} \rightarrow 10{,}000 = 2000\left(1+\frac{0.12}{12}\right)^{12t} \rightarrow$$

$$5 = 1.01^{12t} \rightarrow \ln 5 = 12t\cdot \ln 1.01 \rightarrow$$

$$t = \frac{\ln 5}{12\ln 1.01} \rightarrow t \approx 13.5 \text{ yrs.}$$

48

[illegible]

3.) (10 pts.) The manager of the Economy Motel charges \$30 per room and rents ~~50~~ rooms each night. For each \$5 increase in room charge four (4) fewer rooms are rented. What charge per room will maximize the total amount of money the manager will make in one night ?

Let x : # of \$5 increases,

$$\text{max. } T = (30+5x)(\overset{48}{[illegible]} - 4x) \rightarrow$$

$$\underset{\text{charge per room}}{\uparrow} \qquad \underset{\text{\# of rooms}}{\uparrow}$$

$$T' = (30+5x)(-4) + (5)(\overset{48}{[illegible]} - 4x)$$

$$= -120 - 20x + \underset{240}{[illegible]} - 20x$$

$$= 120 - 40x = 0$$

$$\overset{+ \quad 0 \quad -}{\underset{x=3}{\longrightarrow}} \; T'$$

charge : \$45

rooms : 36

\$: 1620

4.) (10 pts.) Use the limit definition of derivative to differentiate $f(x) = \frac{x^2}{x+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)^2}{x+h+1} - \frac{x^2}{x+1}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2)(x+1) - (x+h+1)x^2}{(x+h+1)\ (x+1)} \cdot \frac{1}{h}$$

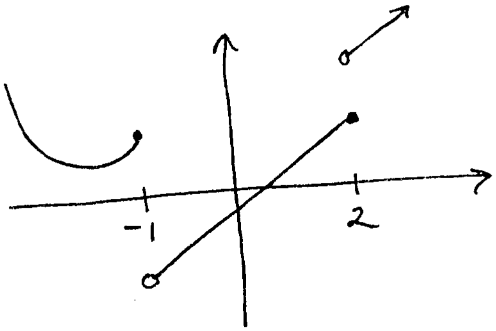
$$= \lim_{h\to 0} \frac{\cancel{x^3} + 2hx^2 + h^2x + \cancel{x^2} + 2hx + h^2 - \cancel{x^3} - hx^2 - \cancel{x^2}}{(x+h+1)(x+1)\ \cdot\ h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x^2 + hx + 2x + h - x^2)}{(x+h+1)\ (x+1) \cdot \cancel{h}}$$

$$= \frac{x^2 + 2x}{(x+1)^2}$$

5.) (10 pts.) Use limits to determine the values of the constants A and B so that the following function is continuous for all values of x .

$$f(x) = \begin{cases} Bx^2 + Ax, & \text{if } x \le -1 \\ 2B - Ax, & \text{if } -1 < x \le 2 \\ x + 3, & \text{if } x > 2 \ . \end{cases}$$



$$\lim_{x\to -1^-} (Bx^2 + Ax) = \lim_{x\to -1^+} (2B - Ax)$$

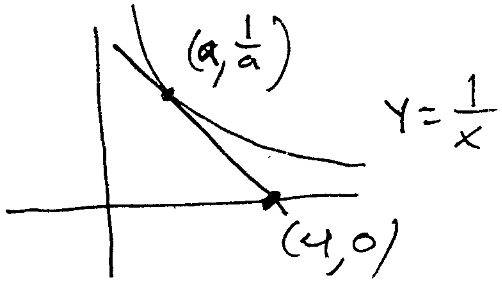
$$\to B - A = 2B + A \to \boxed{B = -2A} \ ;$$

$$\lim_{x\to 2^-} (2B - Ax) = \lim_{x\to 2^+} (x + 3)$$

$$\to \boxed{2B - 2A = 5} \to 2(-2A) - 2A = 5 \to$$

$$-6A = 5 \to \boxed{A = -5/6} \ , \ \boxed{B = 5/3}$$

6.) (10 pts.) Find all points (x, y) on the graph of $y = \frac{1}{x}$ with tangent lines passing through the point (4, 0).



$$Y' = \frac{-1}{x^2}$$

$$\text{slope}: \frac{\frac{1}{a} - 0}{a - 4} = \frac{-1}{a^2} \to$$

$$a = -a + 4 \to a = 2 \text{ so pt. is}$$

$$\left(2, \tfrac{1}{2}\right).$$

7.) (10 pts.) The radius of a sphere is measured with an absolute percentage error of at most 4%. Use differentials to estimate the maximum absolute percentage error in computing the volume of the sphere. ($V = \frac{4}{3}\pi r^3$.)

$V' = 4\pi r^2$ and $\frac{|\Delta r|}{r} \le 4\%$, estimate

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta r|}{V} = \frac{|\cancel{4}\cancel{\pi}\cancel{r^2} \cdot \Delta r|}{\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}}$$

$$= 3\frac{|\Delta r|}{r} \le 3(4\%) = 12\% .$$

8.) Consider the equation $27 - x^3 = \sin x$.

a.) (10 pts.) Use the Intermediate Value Theorem to verify that the equation is solvable. Let $f(x) = 27 - x^3 - \sin x = 0$ and $\leftarrow m = 0$;

f is cont. since difference of continuous functions; consider interval $[2, \pi]$: $f(2) = 19 - \sin 2 > 0$ and $f(\pi) = 27 - \pi^3 - \overset{0}{\sin \pi} < 0$ so $m = 0$ is between $f(2)$ and $f(\pi)$. Thus, by IMVT there is at least one value c, $2 \le c \le \pi$, satisfying $f(c) = 0$, i.e.,

$$27 - c^3 - \sin c = 0 .$$

b.) (5 pts.) Use Newton's method to estimate the value of the solution of the equation to three decimal places. $f'(x) = -3x^2 - \cos x$ so

$$x_{n+1} = x_n - \frac{27 - x_n^3 - \sin x_n}{-3x_n^2 - \cos x_n} = \frac{\sin x_n - x_n \cos x_n - 2x_n^3 - 27}{-3x_n^2 - \cos x_n}$$

$x_1 = 3$

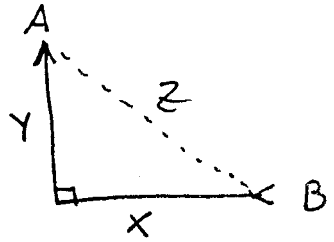
$x_2 = 2.99457$

$x_3 = 2.99456$

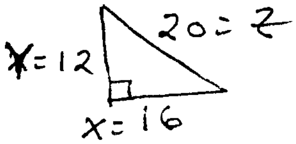
9.) Car B is 34 miles directly east of car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph.

a.) (10 pts.) At what rate is the distance between the cars changing after $t = \frac{1}{5}$ hr. ?

$\frac{dY}{dt} = 60$ mph, $\frac{dx}{dt} = -90$ mph, find $\frac{dz}{dt}$ when $t = \frac{1}{5}$ hr $\rightarrow x = 12$ mi, $Y = 16$ mi. :



$$x^2 + Y^2 = z^2 \rightarrow \cancel{2}x \frac{dx}{dt} + \cancel{2}Y \frac{dY}{dt} = \cancel{2}z \frac{dz}{dt}$$

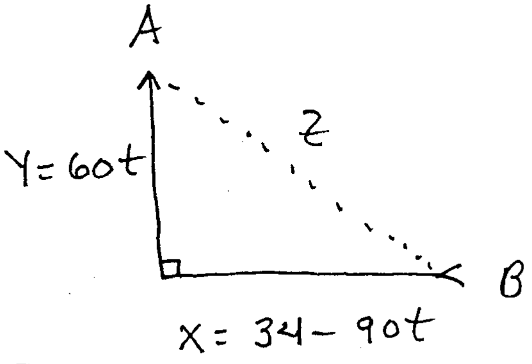


$$\rightarrow (16)(-9\cancel{0}) + (12)(6\cancel{0}) = (2\cancel{0}) \frac{dz}{dt}$$

$$\rightarrow \frac{dz}{dt} = -36 \text{ mph}$$

b.) (10 pts.) What is the minimum distance between the cars and at what time t does the minimum distance occur ?

Let t be time, minimize distance



$$z = \sqrt{(60t)^2 + (34-90t)^2} \rightarrow$$

$$\frac{dz}{dt} = \frac{1}{2}(\sim)^{-\frac{1}{2}} \cdot [\cancel{2}(60t)6\cancel{0} + \cancel{2}(34-90t) \cdot (-9\cancel{0})] = 0 \rightarrow$$

$$360t - 306 + 810t = 0 \longrightarrow t \approx 0.26 \text{ hrs.}$$

$$\underset{t=0.26}{\overset{-\quad 0 \quad +}{\longleftrightarrow}} \; z'$$

and min. $z \approx 18.86$ mi.

10.) (10 pts.) Find the slope and concavity of the graph $xy + y^2 = 3x + 1$ at the point $(0, -1)$.

$$\xrightarrow{D} XY' + Y + 2YY' = 3 \rightarrow Y' = \frac{3-Y}{X+2Y}$$

and $X=0,\ Y=-1 \rightarrow Y' = \frac{4}{-2} = \boxed{-2 = \text{slope}}$;

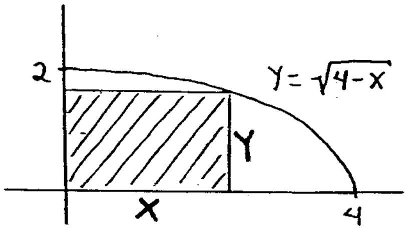
$$Y'' = \frac{(X+2Y)(-Y') - (3-Y)(1+2Y')}{(X+2Y)^2}$$

$$= \frac{(-2)(2) - (4)(-3)}{(-2)^2} = \boxed{2 = Y''}$$ so

concave up.

11.) (10 pts.) Consider all rectangles in the first quadrant inscribed in such a way that their bases lie on the x-axis with the top corner on the graph of $y = \sqrt{4-x}$. Find the length and width of the rectangle of maximum area.

max. area



$$A = XY = X\sqrt{4-X} \longrightarrow$$

$$A' = X \cdot \frac{1}{2}(4-x)^{-1/2}(-1) + \sqrt{4-x}$$

$$= \frac{-x}{2\sqrt{4-x}} + \frac{\sqrt{4-x}}{1} = \frac{-x + 2(4-x)}{2\sqrt{4-x}}$$

$$= \frac{8-3x}{2\sqrt{4-x}} = 0$$

$+ \quad 0 \quad -$ $\quad A'$

$x = 8/3$

$A = \frac{16}{3\sqrt{3}}$ $\quad Y = \frac{2}{\sqrt{3}}$

12.) (15 pts.) Consider the function $f(x) = x\, e^{(\frac{-x}{2})}$. Determine where f is increasing, decreasing, concave up, and concave down. Identify all relative and absolute extrema, inflection points, x- and y-intercepts, and vertical and horizontal asymptotes. Sketch the graph. You may assume that $f'(x) = (1 - \frac{x}{2})\, e^{(\frac{-x}{2})}$ and $f''(x) = (\frac{x}{4} - 1)\, e^{(\frac{-x}{2})}$.

$$\overset{+ \qquad 0 \qquad -}{\overline{\qquad\qquad | \qquad\qquad}}\; f' \qquad\qquad \overset{- \qquad 0 \qquad +}{\overline{\qquad\qquad | \qquad\qquad}}\; f''$$

abs. max. $\begin{cases} X = 2 \\ Y = \frac{2}{e} \end{cases}$ infl. pt. $\begin{cases} X = 4 \\ Y = \frac{4}{e^2} \end{cases}$

$X = 0 : Y = 0$ and $Y = 0 : X = 0$

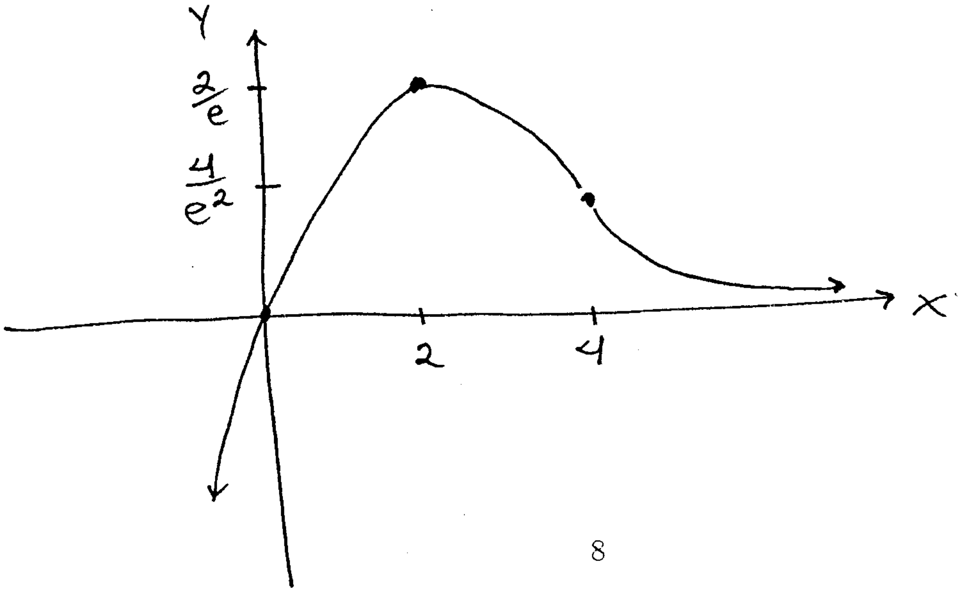
f is ↑ for $X < 2$ f is $\cup$ for $X > 4$

f is ↓ for $X > 2$ f is $\cap$ for $X < 4$

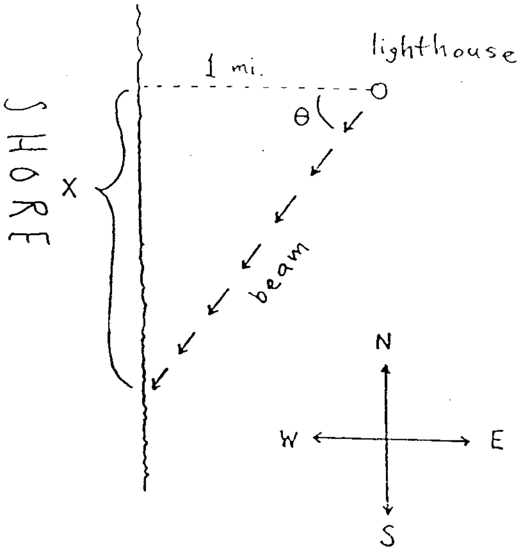
$$\lim_{X \to -\infty} X e^{-\frac{X}{2}} = -\infty \cdot \infty = -\infty$$

$$\lim_{X \to +\infty} X e^{-\frac{X}{2}} = \lim_{X \to +\infty} \frac{X}{e^{X/2}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{X \to +\infty} \frac{1}{\frac{1}{2} e^{X/2}} = \text{"}\frac{1}{\infty}\text{"} = 0$$

so horizontal asymptote $Y = 0$



13.) (10 pts.) A lighthouse sits one (1) mile offshore with a light beam turning counter-clockwise at the rate of ten (10) revolutions per minute. How fast is the light beam racing down the shoreline when the beam strikes a point on the shore twelve miles south of the nearest point on the shore ? ~~Assume there are 5280 feet per mile and~~ write your final answer in MILES PER HOUR.

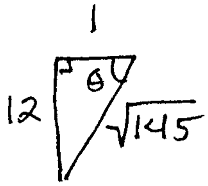


$$\frac{d\theta}{dt} = \frac{10\,\cancel{\text{rev}}.}{\text{min}.} \cdot \frac{2\pi\text{ rad}.}{\cancel{\text{rev}}.} = \frac{20\pi\text{ rad}.}{\text{min}.}\,)$$

find $\frac{dx}{dt}$ when $x = 12$ mi :

$$\tan\theta = \frac{x}{1} = x \xrightarrow{D_t}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$



$$\rightarrow (\sqrt{145})^2 \cdot (20\pi) = \frac{dx}{dt} \rightarrow$$

$$\frac{dx}{dt} = 9110.6\,\frac{\cancel{\text{mi}}.}{\cancel{\text{min}}.} \cdot \frac{60\text{ min}.}{1\text{ hr}.}$$ [redacted]

$= 546{,}660$ mph

[redacted]

14.) (10 pts. each) Evaluate the following limits.

a.) $\lim_{x\to 0} \frac{x \sin x}{(\arctan x)^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{x\cos x + \sin x}{2\arctan x \cdot \frac{1}{1+x^2}}$

$$= \lim_{x\to 0} \frac{(x\cos x + \sin x)(x^2+1)}{2\arctan x}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{(x\cos x + \sin x)(2x) + (-x\sin x + \cos x + \cos x)(x^2+1)}{2 \cdot \frac{1}{1+x^2}}$$

$$= \frac{2}{2} = 1$$

b.) $\lim\limits_{x\to0+} x\ln x$ $= \text{"}0\cdot-\infty\text{"} = \lim\limits_{x\to0^+} \dfrac{\ln x}{\frac{1}{x}}$

$$\overset{\text{"}\frac{-\infty}{\infty}\text{"}}{=} \lim_{x\to0+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to0+}(-x) = 0$$

c.) $\lim\limits_{x\to\infty} (x^3+4)^{\frac{1}{x}}$ $= \text{"}\infty^0\text{"}$

$$\ln\left(\lim_{x\to\infty}(x^3+4)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \ln(x^3+4)^{1/x}$$

$$= \lim_{x\to\infty} \frac{\ln(x^3+4)}{x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{\frac{3x^2}{x^3+4}}{1}$$

$$= \lim_{x\to\infty} \frac{3x^2\cdot\frac{1}{x^2}}{(x^3+4)\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3}{x+\frac{4}{x^2}}$$

$$= \text{"}\frac{3}{\infty+0}\text{"} = 0 \quad \text{so} \quad \lim_{x\to\infty}(x^3+4)^{1/x} = 1$$

Each of the following three EXTRA CREDIT PROBLEMS is worth 10 points. These problems are OPTIONAL.

1.) Show that $\log_B C = \dfrac{\ln C}{\ln B}$.

Let $\log_B C = X \rightarrow$

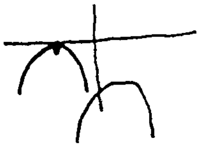
$B^X = C \rightarrow \ln B^X = \ln C \rightarrow X \ln B = \ln C \rightarrow$

$X = \dfrac{\ln C}{\ln B} \rightarrow \log_B C = \dfrac{\ln C}{\ln B}$

2.) Find all values of K for which the the function $\underline{f(x) = \boxed{-x^3} + Kx + x^2}$ is NOT one-to-one.

f $\underline{\underline{\underline{\text{is}}}}$ 1-1 if $f'(x) \leq 0 \rightarrow f'(x) = -3x^2 + K + 2X \leq 0 \rightarrow$

$-3x^2 + 2x + K \leq 0$ $\qquad x = \dfrac{-2 \pm \sqrt{4 + 12K}}{-6}$



so $4 + 12K \leq 0 \rightarrow K \leq \frac{-1}{3}$

NOT 1-1 : $K > \frac{-1}{3}$

3.) Find a tilted asymptote for the function $y = \sqrt{x^2 + x}$.

Since $\sqrt{x^2+x} = \sqrt{(x+\frac{1}{2})^2 - \frac{1}{4}}$, conjecture that either

$$Y = x + \frac{1}{2} \quad \text{or} \quad Y = -(x+\frac{1}{2})$$

is a tilted asymptote. Solving

$$\sqrt{x^2+x} = x + \frac{1}{2} + \frac{1}{f(x)}$$ for $f(x)$ leads to

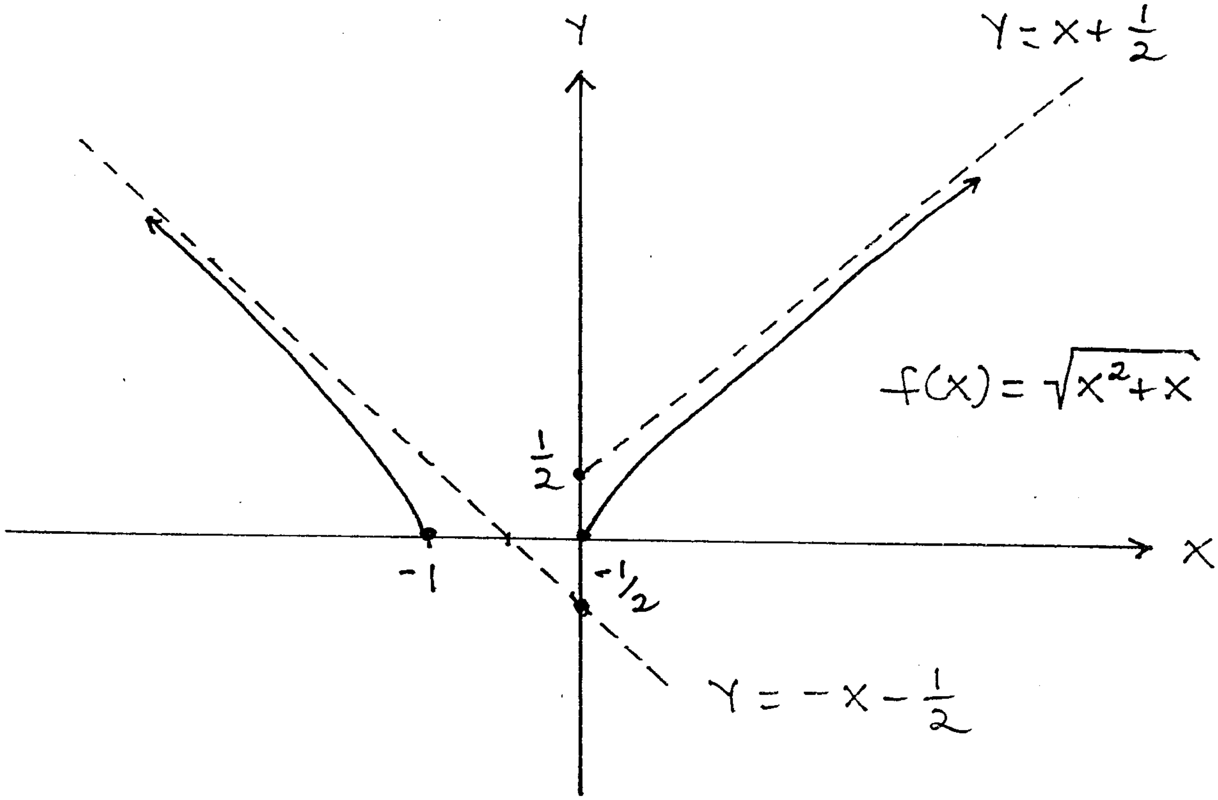
(1) $$\sqrt{x^2+x} = x + \frac{1}{2} + \frac{\frac{-1}{4}}{\sqrt{x^2+x} + (x+\frac{1}{2})} .$$

Since $\lim_{x \to \infty} \frac{\frac{-1}{4}}{\sqrt{x^2+x} + (x+\frac{1}{2})} = 0$, $Y = x + \frac{1}{2}$ is a tilted asymptote for $f(x) = \sqrt{x^2+x}$. Similarly,

(2) $$\sqrt{x^2+x} = -(x+\frac{1}{2}) + \frac{\frac{-1}{4}}{\sqrt{x^2+x} - (x+\frac{1}{2})} .$$

Since $\lim_{x \to -\infty} \frac{\frac{-1}{4}}{\sqrt{x^2+x} - (x+\frac{1}{2})} = 0$, $Y = -(x+\frac{1}{2})$ is a tilted asymptote for $f(x) = \sqrt{x^2+x}$.

SEE GRAPH ON NEXT PAGE .



Y
$Y = x + \frac{1}{2}$
$f(x) = \sqrt{x^2 + x}$
$\frac{1}{2}$
X
-1
$-1/2$
$Y = -x - \frac{1}{2}$