Math 21A
Kouba
Exam 1

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 6 pages, including the cover page.

6. You may NOT use L'Hopital's Rule on this exam.

7. You may NOT use the shortcut for finding limits to infinity.

8. Using only a calculator to determine limits will receive little or no credit.

9. You will be graded on proper use of limit notation.

10. You have until 9:50 a.m. sharp to finish the exam.

1.) (9 pts. each) Determine the following limits.

a.) $\lim_{x \to 2} \dfrac{x-2}{x^2+x-6} = \text{"}\dfrac{0}{0}\text{"} = \lim_{x \to 2} \dfrac{\cancel{(x-2)}}{\cancel{(x-2)}(x+3)} = \dfrac{1}{5}$

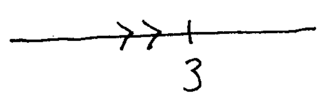
b.) $\lim_{x \to 0} \dfrac{\frac{1}{x+1} - 1}{\frac{x}{1}} = \text{"}\dfrac{0}{0}\text{"} = \lim_{x \to 0} \dfrac{1-(x+1)}{x+1} \cdot \dfrac{1}{x}$

$= \lim_{x \to 0} \dfrac{-\cancel{x}}{(x+1)\cancel{x}} = \dfrac{-1}{1} = -1$

c.) $\lim_{h \to 0} \dfrac{\sin h^2}{h} = \lim_{h \to 0} \dfrac{h}{h} \cdot \dfrac{\sin h^2}{h} = \lim_{h \to 0} h \cdot \dfrac{\sin h^2}{h^2}$

$= 0 \cdot 1 = 0$

d.) $\lim_{x \to +3^-} \dfrac{x^2-5}{3-x} = \dfrac{4}{0^+} = +\infty$



e.) $\lim_{x \to 1} \dfrac{2-\sqrt{x+3}}{x-1} = \text{"}\dfrac{0}{0}\text{"} = \lim_{x \to 1} \dfrac{2-\sqrt{x+3}}{x-1} \cdot \dfrac{2+\sqrt{x+3}}{2+\sqrt{x+3}}$

$= \lim_{x \to 1} \dfrac{4-(x+3)}{(x-1)(2+\sqrt{x+3})} = \lim_{x \to 1} \dfrac{-\cancel{(x-1)}}{\cancel{(x-1)}(2+\sqrt{x+3})} = \dfrac{-1}{4}$

f.) $\lim_{x \to \infty} \dfrac{\cos(3x+1)}{3x+1}$ (HINT : Use the Squeeze Principle.) $-1 \le \cos(3x+1) \le +1$

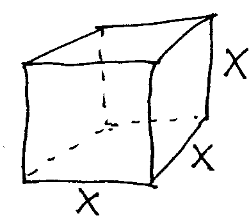
$\rightarrow \dfrac{-1}{3x+1} \le \dfrac{\cos(3x+1)}{3x+1} \le \dfrac{+1}{3x+1}$ and $\lim_{x \to \infty} \dfrac{-1}{3x+1} = 0 = \lim_{x \to \infty} \dfrac{1}{3x+1}$

so $\lim_{x \to \infty} \dfrac{\cos(3x+1)}{3x+1} = 0$

2.) (8 pts.) Determine the domain for $f(x) = \frac{3}{4 - \sqrt{x}}$.

$X \geq 6$ and $\sqrt{x} \neq 4$ so $X \neq 16$ so

Domain : all $X \geq 0$ except $X = 16$



3.) Consider a three-dimensional cube with side length x.

a.) (2 pts.) Write the volume V of the cube as a function of x.

$V = x^3$

b.) (2 pts.) Write the surface area S of the cube as a function of x.

$S = 6x^2$

c.) (4 pts.) Write the surface area S of the cube as a function of the volume V.

$V = x^3 \rightarrow \quad x = V^{1/3} \quad \rightarrow$

$S = 6x^2 = 6\left(V^{1/3}\right)^2 = 6V^{2/3}$.

4.) Consider the following function $f(x) = \begin{cases} \dfrac{x^2-3x}{x^2-9}, & \text{if } x \neq 3, -3 \\ \frac{1}{2}, & \text{if } x = 3 \\ 0, & \text{if } x = -3 . \end{cases}$

(6 pts.) Determine if f is continuous at $x = 3$.

i.) $f(3) = \frac{1}{2}$

ii.) $\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{x^2-3x}{x^2-9} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 3} \dfrac{x\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$

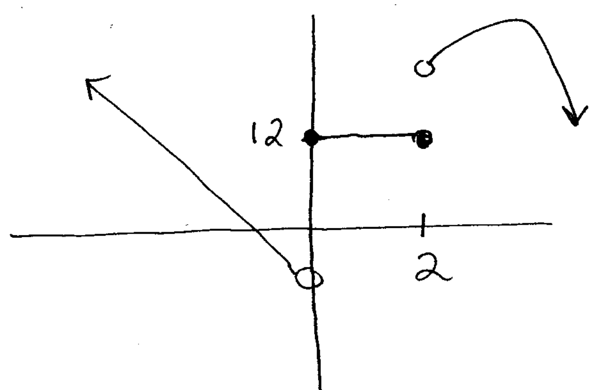
$= 3/6 = 1/2$

iii.) $\lim_{x \to 3} f(x) = f(3)$ so

f is continuous at $x = 3$.

5.) (6 pts.) Using limits, determine the value(s) of constants A and B so that the following function is continuous for all values of x :

$$f(x) = \begin{cases} Ax + B, & \text{if } x < 0 \\ 12 , & \text{if } 0 \leq x \leq 2 \\ Bx^2 - A, & \text{if } x > 2 . \end{cases}$$



$\underline{\lim_{x \to 0^-} (Ax+B) = 12}$ and $\underline{\lim_{x \to 2^+} (Bx^2 - A) = 12}$

$\rightarrow$ $B = 12$ and $4B - A = 12$ $\rightarrow$

$48 - A = 12$ $\rightarrow$ $A = 36$

6.) (9 pts.) Use the Intermediate Value Theorem to prove that the equation $x^3 = x^2 + 5$ is solvable. This is a writing exercise.

Since $x^3 - x^2 = 5$ let $f(x) = x^3 - x^2$ and $m = 5$, where f is continuous for all x-values since it is a polynomial. Since $f(0) = 0$ and $f(3) = 18$ and $m = 5$ is between $f(0)$ and $f(3)$, choose interval $[0, 3]$. By the IMVT there is at least one number c, $0 \le c \le 3$, so that $f(c) = m$, i.e., $c^3 - c^2 = 5$ and equation is solvable.

7.) (9 pts.) Give an ε, δ-proof for the following limit. This is a writing exercise. : $\lim_{x \to -1} (x^2 + 3) = 4$

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x+1| < \delta$, then $|f(x) - 4| < \varepsilon$. Begin with $|f(x) - 4| < \varepsilon$ and solve for $|x+1|$.

Then $|f(x) - 4| < \varepsilon$

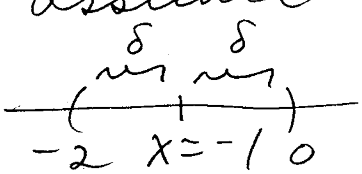
iff $|(x^2 + 3) - 4| < \varepsilon$

iff $|x^2 - 1| < \varepsilon$

iff $|x-1||x+1| < \varepsilon$.

Replace $|x-1|$.

Assume $\delta \le 1$



so that $-2 < x < 0$ and $1 < |x-1| < 3$. Then $|x-1||x+1| < 3|x+1| < \varepsilon$ iff $3|x+1| < \varepsilon$ iff $|x+1| < \varepsilon/3$. Choose $\delta = \min\{\varepsilon/3, 1\}$.

Thus, if $0 < |x+1| < \delta$, it follows that $|f(x) - 4| < \varepsilon$.

The following EXTRA CREDIT PROBLEM is worth 10 points. This problems is OPTIONAL.

1.) Determine the following limit : $\lim_{x \to \infty} (x - \sqrt{x^2 + 9x}\,) = \infty - \infty$

$$\lim_{x \to \infty} (x - \sqrt{x^2 + 9x}) \frac{(x + \sqrt{x^2 + 9x})}{(x + \sqrt{x^2 + 9x})}$$

$$= \lim_{x \to \infty} \frac{x^2 - (x^2 + 9x)}{x + \sqrt{x^2 + 9x}}$$

$$= \lim_{x \to \infty} \frac{-9x}{x + \sqrt{x^2 + 9x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{-9}{1 + \sqrt{x^2 + 9x} \cdot \sqrt{\frac{1}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{-9}{1 + \sqrt{1 + 9/x}} =$$

$$= \frac{-9}{1 + \sqrt{1 + 0}}$$

$$= -\frac{9}{2}.$$