Math 21A
Kouba
Exam 2

KEY

Please PRINT your name here : ____

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 6 pages, including the cover page.

6. You may NOT use L'Hopital's Rule on this exam.

7. You may NOT use the shortcut for finding limits to infinity.

8. Using only a calculator to determine limits will receive little credit.

9. You will be graded on proper use of limit and derivative notation.

10. You have until 9:50 am to finish the exam.

1.) (7 pts. each) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

a.) $y = x^2 \sin(4x)$

$$y' = x^2 \cdot \cos(4x) \cdot 4 + 2x \cdot \sin(4x)$$

b.) $f(x) = \dfrac{3x+7}{\sqrt{x^4+1}}$

$$f'(x) = \frac{\sqrt{x^4+1} \cdot (3) - (3x+7) \cdot \frac{1}{2}(x^4+1)^{-1/2} \cdot 4x^3}{x^4+1}$$

c.) $g(x) = \sec^3(\tan^{-4}(x^{1/5}))$

$$g'(x) = 3\sec^2(\tan^{-4}(x^{1/5})) \cdot \sec(\tan^{-4}(x^{1/5}))\tan(\tan^{-4}(x^{1/5})) \cdot \to -4\tan^{-5}(x^{1/5}) \cdot \sec^2(x^{1/5}) \cdot \tfrac{1}{5}x^{-4/5}$$

2.) (12 pts.) Use $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to differentiate the function $f(x) = \dfrac{x}{x+5}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h+5} - \frac{x}{x+5}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{(x+h)(x+5) - x(x+h+5)}{(x+h+5)(x+5)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{5x} + \cancel{hx} + 5h - \cancel{x^2} - \cancel{hx} - \cancel{5x}}{(x+h+5)(x+5) \cdot h}$$

$$= \lim_{h \to 0} \frac{5\cancel{h}}{(x+h+5)(x+5)\cancel{h}}$$

$$= \frac{5}{(x+5)^2}$$

3.) You are standing on the top edge of a building which is 96 ft. high. You throw an apple straight UP at 80 ft./sec. and watch as it falls back to the ground.

a.) (8 pts.) Assume that the acceleration due to gravity is $s''(t) = -32$ ft./sec.^{2} . Derive velocity, $s'(t)$, and height (above ground), $s(t)$, formulas for this apple.

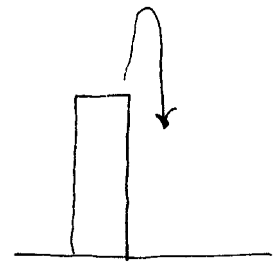
$s''(t) = -32 \rightarrow$

$s'(t) = -32t + C$ (and $s'(0) = 80$ ft./sec.)

$\rightarrow C = 80 \rightarrow \boxed{s'(t) = -32t + 80}$;

$s(t) = -16t^2 + 80t + C$ (and $s(0) = 96$ ft.)

$\rightarrow C = 96 \rightarrow \boxed{s(t) = -16t^2 + 80t + 96}$



b.) (3 pts.) In how many seconds will the apple strike the ground ?

strike ground : $s(t) = 0 \rightarrow$

$-16t^2 + 80t + 96 = 0 \rightarrow -16(t^2 - 5t - 6) = 0 \rightarrow$

$-16(t-6)(t+1) = 0 \rightarrow$ $t = 6$ sec.

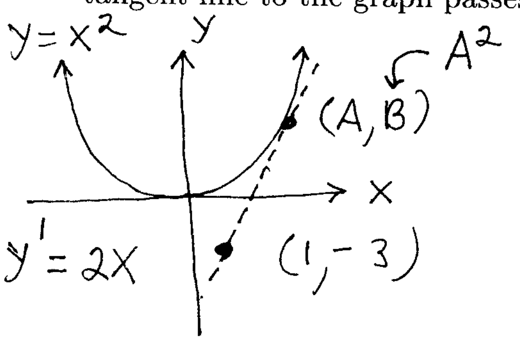
c.) (3 pts.) How high does the apple go ?

highest point : $s'(t) = 0 \rightarrow -32t + 80 = 0 \rightarrow$

$\boxed{t = 2.5 \text{ sec.}} \rightarrow$

$s(2.5) = -16(2.5)^2 + 80(2.5) + 96 =$ 196 ft.

4.) (10 pts.) Find all point(s) (A, B) on the graph of $y = x^2$, so that at those point(s), the tangent line to the graph passes through the point $(1, -3)$. HINT : Draw a sketch.



Slope of tangent at $x = A$ can be found two ways :

I.) slope $= 2A$ (derivative)

II.) slope $= \dfrac{B - (-3)}{A - 1}$ $\left(\dfrac{\text{rise}}{\text{run}}\right)$

$= \dfrac{A^2 + 3}{A - 1}$; thus

$2A = \dfrac{A^2 + 3}{A - 1} \rightarrow 2A^2 - 2A = A^2 + 3 \rightarrow$

$A^2 - 2A - 3 = 0 \rightarrow (A-3)(A+1) = 0 \rightarrow A = 3$ or $A = -1$

so points are $(3, 9)$, $(-1, 1)$.

5.) (15 pts.) Consider the function $f(x) = x(x-4)^3$. Determine where f is increasing, decreasing, concave up, and concave down. Identify all relative and absolute extrema, inflection points, and x- and y-intercepts. Sketch the graph. You may assume that

$$f'(x) = 4(x-4)^2(x-1) \quad \text{and} \quad f''(x) = 12(x-4)(x-2) .$$

$- \quad 0 \quad + \quad 0 \quad +$ $\quad f'$

$x=1 \qquad x=4$

$\underbrace{y=-27}_{\text{abs. min.}} \qquad y=0$

$+ \quad 0 \quad - \quad 0 \quad +$ $\quad f''$

$x=2 \qquad x=4$

$y=-16 \qquad y=0$

Infl. Pts.

f is ↑ for $1<x<4$, $x>4$,

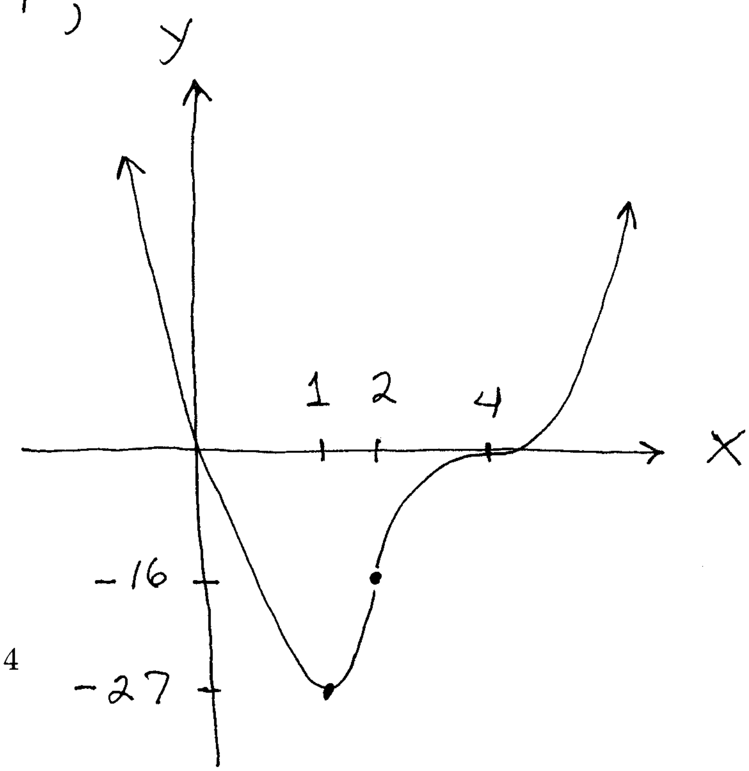
f is ↓ for $x<1$,

f is ∪ for $x<2$, $x>4$,

f is ∩ for $2<x<4$;

$x=0$: $y=0$

$y=0$: $x=0$, $x=4$



6.) (10 pts.) Determine if the following function satisfies the assumptions of the MVT. If so, find all values of c guaranteed by the conclusion of the MVT.

$f(x) = x^{2/3}$ on the interval [0, 8]

$f(x) = x^{2/3} = (x^2)^{1/3}$ is continuous on $[0,8]$ since it is the composition of continuous functions ($y = x^2$, $y = x^{1/3}$); $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$ so f is differentiable on $(0,8)$; thus, by MVT there is at least one value c, $0 < c < 8$, so that $f'(c) = \frac{f(8) - f(0)}{8 - 0} \rightarrow$

$\frac{2}{3c^{1/3}} = \frac{4-0}{8-0} \rightarrow \frac{4}{3} = c^{1/3} \rightarrow c = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$.

7.) (8 pts.) Assume that $f(t)$ is the weight (in tons) of an iceberg at time t (in years). What are the proper units for the derivative, $f'(t)$, and briefly explain what $f'(t)$ represents in this context.

$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$ with units $\frac{\text{tons}}{\text{yr.}}$

represents the rate at which the weight of an iceberg changes at time t.

8.) (10 pts.) Assume that $y = g(x)$ is a differentiable function. Use the limit definition of derivative to derive a formula for the derivative of $[g(x)]^{-1} = \frac{1}{g(x)}$.

Let $f(x) = \frac{1}{g(x)}$ then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{g(x) - g(x+h)}{g(x+h)\,g(x) \cdot h}$$

$$= \lim_{h \to 0} \frac{-1}{g(x+h)\,g(x)} \cdot \frac{g(x+h) - g(x)}{h} = \frac{-1}{(g(x))^2} \cdot g'(x)$$

The following EXTRA CREDIT PROBLEM is worth 12 points. This problem is OPTIONAL.

1.) Consider the function $f(x) = \begin{cases} x^2\cos(1/x), & \text{if } x \neq 0 \\ 0 \ , & \text{if } x = 0 \ . \end{cases}$

a.) (5 pts.) Determine if f is continuous at $x = 0$.

b.) (7 pts.) Determine if f is differentiable at $x = 0$.

a.) i.) $f(0) = 0$

ii) $-1 \le \cos(1/x) \le +1 \rightarrow -x^2 \le x^2\cos(1/x) \le x^2$

and $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$, so by

Squeeze Principle $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0$.

iii) $f(0) = \lim_{x\to 0} f(x)$ so f is cont. at $x=0$.

b.) $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{h^2\cos(1/h)}{h}$

$= \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$ since:

$-1 \le \cos\left(\frac{1}{h}\right) \le +1 \xrightarrow{h>0} -h \le h\cos\left(\frac{1}{h}\right) \le +h$ or

$\xrightarrow{h<0} -h \ge h\cos\left(\frac{1}{h}\right) \ge +h$; in either case,

$\lim_{h\to 0} h = 0 = \lim_{h\to 0} (-h)$ so by Squeeze

Principle, $\lim_{h\to 0} h\cos(1/h) = 0$