Math 21A
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ___________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

2. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

4. Make sure that you have 6 pages, including the cover page.

5. On optimization (maximum/minimum) problems use a sign chart to verify the minimum or maximum and list optimal values for ALL variables used in the problem.

6. You will be graded on proper use of limit and derivative notation.

7. Put units on answers where units are appropriate.

8. You have until 9:50 a.m. sharp to finish the exam.

1.) (7 pts. each) Determine $y' = \dfrac{dy}{dx}$. DO NOT SIMPLIFY ANSWERS.

a.) $y = \ln(\sin x) + \log(\cos x)$

$$Y' = \frac{1}{\sin x} \cdot \cos x + \frac{1}{\cos x} \cdot -\sin x \cdot \log e$$

b.) $y = (\tan x)^{x^2}$ $\longrightarrow$ $\ln y = \ln(\tan x)^{x^2} = x^2 \ln(\tan x)$

$$\xrightarrow{D} \frac{1}{Y} y' = x^2 \cdot \frac{\sec^2 x}{\tan x} + 2x \cdot \ln(\tan x) \longrightarrow$$

$$Y' = (\tan x)^{x^2} \cdot \left\{ \frac{x^2 \sec^2 x}{\tan x} + 2x \ln(\tan x) \right\}$$

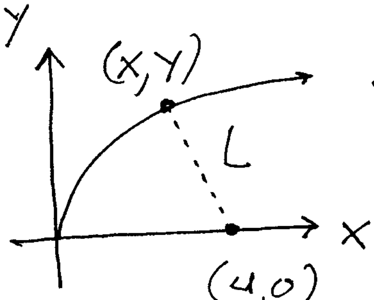
2.) (10 pts.) An initial deposit of \$5000 in an account grows to \$50,000 in t years. If the annual interest rate is 8% and interest is compounded weekly (52 weeks per year), what is t ?

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \longrightarrow 50{,}000 = 5000\left(1 + \frac{0.08}{52}\right)^{52t}$$

$$\longrightarrow 10 = 1.001538462^{52t} \longrightarrow \ln 10 = 52t \ln 1.001538462$$

$$\longrightarrow t = \frac{\ln 10}{52 \ln 1.001538462} \approx 28.8 \text{ yrs.}$$

3.) (11 pts.) Find the point on the graph of $y = 2(\sqrt{x})$ nearest the point $(4, 0)$.



$$\text{Min. } L = \sqrt{(Y-0)^2 + (X-4)^2}$$
$$= \sqrt{Y^2 + (X-4)^2}$$
$$= \sqrt{(2\sqrt{X})^2 + (X-4)^2}$$
$$= \sqrt{4X + (X-4)^2} \xrightarrow{D}$$

$$L' = \frac{1}{2}(\sim)^{-\frac{1}{2}}\{4 + 2(X-4)\} = 0 \to 4 + 2X - 8 = 0 \to$$
$$2X - 4 = 0 \to X = 2$$

$-\quad 0 \quad +$ $\quad L'$

$X = 2,\ Y = 2\sqrt{2}$

and min $\quad L = \sqrt{12} = 2\sqrt{3}$

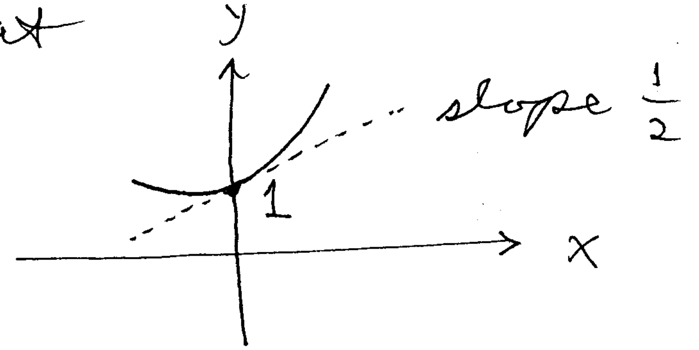
4.) (11 pts.) Assume that y is a function of x . Determine the slope and concavity of the graph of $xy = y^2 - 1$ at $x = 0$, $y = 1$. Sketch the graph near $x = 0$, $y = 1$.

$$XY = Y^2 - 1 \xrightarrow{D} XY' + Y = 2YY' \to Y = 2YY' - XY' \to$$
$$Y' = \frac{Y}{2Y - X} \to \text{slope } Y' = \frac{1}{2}\ ; \text{ then}$$

$$y'' = \frac{(2Y - X)y' - y(2y' - 1)}{(2Y - X)^2} \text{ at } X = 0, y = 1, y' = \tfrac{1}{2} \to$$

$$y'' = \frac{(2)(\frac{1}{2}) - 1(0)}{(2)^2} = \frac{1}{4} \text{ so concavity } y'' = \frac{1}{4};$$

graph ↑ and is ∪ at
$X = 0, Y = 1$:



5.) (10 pts.) Use Newton's Method to estimate the value of $(20)^{1/4}$ to three decimal places. HINT : Apply Newton's Method to the equation $x^4 - 20 = 0$. Let $x_1 = 2$.

$$f(x) = x^4 - 20 \rightarrow f'(x) = 4x^3 \rightarrow$$

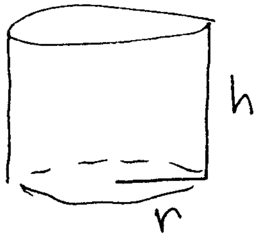
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 20}{4x_n^3} = \frac{3x_n^4 + 20}{4x_n^3} \rightarrow$$

$$x_{n+1} = \frac{3x_n^4 + 20}{4x_n^3} \; ; \quad x_1 = 2 \rightarrow$$

$$x_2 = \frac{68}{32} = 2.125 \rightarrow x_3 \approx 2.114 \rightarrow$$

$x_4 \approx 2.114$ so $20^{1/4} \approx 2.114$

6.) (11 pts.) A cylindrical can with no top is to have a volume of 8π in.^{3} What radius r and height h will result in a can of minimum surface area ? RECALL : The volume of a cylinder is $V = \pi r^2 h$.



volume $\cancel{\pi} r^2 h = 8\cancel{\pi} \rightarrow h = 8/r^2$;

max. surface area

$$S = \pi r^2 + 2\pi r h \rightarrow$$

$$S = \pi r^2 + 2\pi r\left(\frac{8}{r^2}\right) = \pi r^2 + \frac{16\pi}{r} \quad \text{then}$$

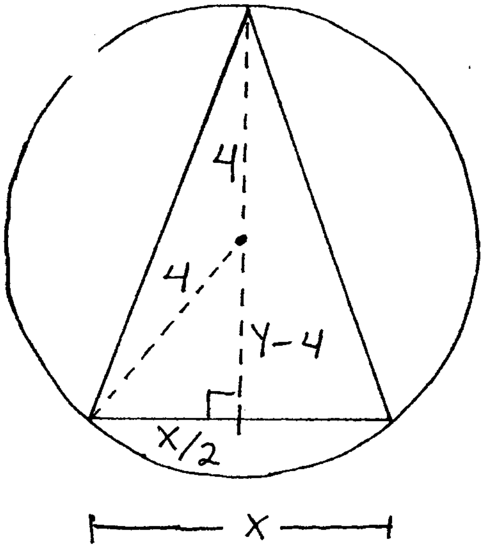
$$S' = 2\pi r - \frac{16\pi}{r^2} = \frac{2\pi r^3 - 16\pi}{r^2} = \frac{2\pi(r^3 - 8)}{r^2} = 0$$

$\rightarrow r = 2$ in.

$$\begin{array}{ccc} - & 0 & + \\ \hline & r = 2 \text{ in.} & \end{array} \quad S'$$

$r = 2$ in.
$h = 2$ in.
and min. $S = 12\pi$ in.^{3}

7.) (11 pts.) Determine the height y and the base x of the isosceles triangle (2 edges are equal) of largest area which can be inscribed in a circle of radius 4 inches.



$$\left(\frac{X}{2}\right)^2 + (Y-4)^2 = 4^2 \to$$

$$\frac{X^2}{4} = 16 - (Y^2 - 8Y + 16) \to$$

$$X^2/4 = 8Y - Y^2 \to$$

$$X^2 = 32Y - 4Y^2 \to X = \sqrt{32Y - 4Y^2} = 2\sqrt{8Y - Y^2};$$

max. area $A = \frac{1}{2} XY \to$

$$A = \frac{1}{2} Y \cdot 2\sqrt{8Y - Y^2} = Y\sqrt{8Y - Y^2}$$

$$\xrightarrow{D} A' = Y \cdot \frac{1}{2}(8Y - Y^2)^{-1/2} \cdot (8 - 2Y) + 1 \cdot \sqrt{8Y - Y^2}$$

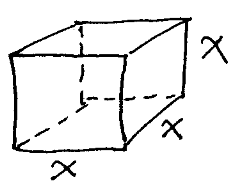
$$= \frac{Y(4-Y)}{\sqrt{8Y - Y^2}} + \frac{\sqrt{8Y - Y^2}}{1} = \frac{4Y - Y^2 + 8Y - Y^2}{\sqrt{8Y - Y^2}} = 0 \to 12Y - 2Y^2 = 0 \to 2Y(6 - Y) = 0$$

$$\begin{array}{ccc} + & 0 & - \end{array} \quad A'$$

$Y = 6''$, $X = 2\sqrt{12}''$

$A = 6\sqrt{12}$ in

8.) (10 pts.) The edge x of a cube is measured with an absolute percentage error of at most 5% . Use differentials to estimate the maximum absolute percentage error in computing the cube's surface area.



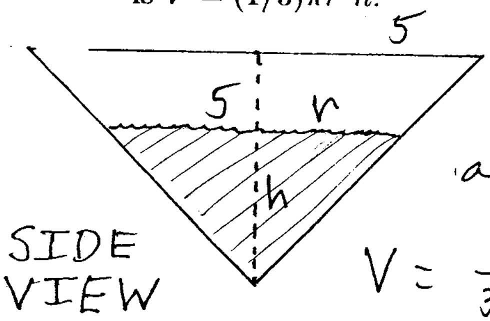
Given $\left|\frac{\Delta X}{X}\right| \leq 5\%$, find $\left|\frac{\Delta A}{A}\right|$;

surface area $A = 6X^2 \to A' = 12X$ so

$$\left|\frac{\Delta A}{A}\right| \approx \left|\frac{dA}{A}\right| = \left|\frac{A' \cdot \Delta X}{6X^2}\right| = \left|\frac{12X \cdot \Delta X}{6X^2}\right|$$

$$= 2\left|\frac{\Delta X}{X}\right| \leq 2(5\%) = 10\%$$

9.) (12 pts.) A tank in the shape of a cone (vertex down) of height 5 ft. and radius 5 ft. is full of water. If water is removed from the tank at the rate of 8π ft.^{3}/min., at what rate is the circular surface area of the water changing ? when $r = 4$ ft. RECALL : The volume of a cone is $V = (1/3)\pi r^2 h$.



$\frac{r}{h} = \frac{5}{5} = 1 \rightarrow r = h$;

assume $\frac{dV}{dt} = -8\pi$ ft.^{3}/min. ;

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \xrightarrow{D}$

$\frac{dV}{dt} = \frac{1}{3}\pi \cdot 3r^2 \frac{dr}{dt} \rightarrow -8\pi = \pi(4)^2 \cdot \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{-1}{2} \frac{\text{ft.}}{\text{min}}$

Find $\frac{dA}{dt}$: $A = \pi r^2 \rightarrow$

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi(4)\left(\frac{-1}{2}\right) = -4\pi$ ft.^{2}/min.

The following EXTRA CREDIT PROBLEM is worth 10 points. It is OPTIONAL.

1.) Use differentials to show that $\sqrt{4+h^2} \approx 2 + (1/4)h^2$ for small h.

Let $f(x) = \sqrt{x}$ and $x : 4 \rightarrow 4 + h^2 \rightarrow$

$f'(x) = \frac{1}{2\sqrt{x}}$, $\Delta x = h^2$; then

$\Delta f = f(4+h^2) - f(4) = \sqrt{4+h^2} - \sqrt{4}$ and

$df = f'(4) \cdot \Delta x = \frac{1}{4}h^2 \rightarrow \Delta f \approx df$ so

$\sqrt{4+h^2} - 2 \approx \frac{1}{4}h^2 \rightarrow$

$\sqrt{4+h^2} \approx 2 + \frac{1}{4}h^2$.