

Math 21A
Kouba
Proving the Derivative of Logarithm

Let $f(x) = \log_b(x)$. Then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\log_b(x+h) - \log_b(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \log_b\left(\frac{x+h}{x}\right) \quad (\text{by property of logarithms}) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \log_b(1+h/x) \\ &= \lim_{h \rightarrow 0} \log_b(1+h/x)^{1/h} \quad (\text{by property of logarithms}) \\ &= \lim_{h \rightarrow 0} \log_b \left[(1+h/x)^{1/(h/x)} \right]^{(1/x)} \\ &= \log_b[e]^{(1/x)} \quad (\text{by definition of } e) \\ &= \frac{1}{x} \log_b[e] \quad (\text{by property of logarithms}) \end{aligned}$$

If $b = 10$, then $D \log x = \frac{1}{x} \log e$.

If $b = e$, then $D \ln x = \frac{1}{x}$.