Section 3.7

1.) $X^2Y + XY^2 = 6 \xrightarrow{D}$

$X^2 \cdot Y' + 2X \cdot Y + X \cdot 2YY' + (1)\cdot Y^2 = 0 \rightarrow$

$Y'(X^2 + 2XY) = -2XY - Y^2 \rightarrow$

$$Y' = \frac{-2XY - Y^2}{X^2 + 2XY}$$

4.) $X^3 - XY + Y^3 = 1 \xrightarrow{D}$

$3X^2 - (X \cdot Y' + (1)\cdot Y) + 3Y^2 \cdot Y' = 0 \rightarrow$

$3X^2 - XY' - Y + 3Y^2Y' = 0 \rightarrow$

$Y'(3Y^2 - X) = Y - 3X^2 \rightarrow$

$$Y' = \frac{Y - 3X^2}{3Y^2 - X}$$

6.) $(3XY+7)^2 = 6Y \xrightarrow{D}$

$\cancel{2}(3XY+7)\cdot(3X\cdot Y' + 3\cdot Y) = \overset{3}{\cancel{6}}Y' \rightarrow$

$9X^2YY' + 21XY' + 9XY^2 + 21Y = 3Y' \rightarrow$

$9X^2YY' + 21XY' - 3Y' = -9XY^2 - 21Y \rightarrow$

$\cancel{3}Y'(3X^2Y + 7X - 1) = \cancel{3}(-3XY^2 - 7Y) \rightarrow$

$$Y' = \frac{-3XY^2 - 7Y}{3X^2Y + 7X - 1}$$

9.) $X = \sec Y \xrightarrow{D} 1 = \sec Y \tan Y \cdot Y' \rightarrow$

$$Y' = \frac{1}{\sec Y \tan Y}$$

11.) $X = \tan(XY) \xrightarrow{D} 1 = \sec^2(XY)\cdot(XY' + (1)Y)$

$\rightarrow 1 = XY'\sec^2(XY) + Y\sec^2(XY)$

$\rightarrow \quad XY' \sec^2(XY) = 1 - Y\sec^2(XY)$

$$\rightarrow \quad Y' = \frac{1 - Y\sec^2(XY)}{X\sec^2(XY)}$$

12.) $X^4 + \sin Y = X^3 Y^2 \xrightarrow{D}$

$4X^3 + \cos Y \cdot Y' = X^3 \cdot 2YY' + 3X^2 \cdot Y^2 \rightarrow$

$\cos Y \cdot Y' - 2X^3 YY' = 3X^2Y^2 - 4X^3 \rightarrow$

$Y'(\cos Y - 2X^3Y) = 3X^2Y^2 - 4X^3 \rightarrow$

$$Y' = \frac{3X^2Y^2 - 4X^3}{\cos Y - 2X^3Y}$$

15.) $e^{2X} = \sin(X + 3Y) \xrightarrow{D}$

$2e^{2X} = \cos(X+3Y) \cdot (1 + 3Y') \rightarrow$

$2e^{2X} = \cos(X+3Y) + 3\cos(X+3Y) \cdot Y' \rightarrow$

$3\cos(X+3Y) \cdot Y' = 2e^{2X} - \cos(X+3Y) \rightarrow$

$$Y' = \frac{2e^{2X} - \cos(X+3Y)}{3\cos(X+3Y)}$$

16.) $e^{X^2Y} = 2X + 2Y \xrightarrow{D}$

$e^{X^2Y} \cdot (X^2 \cdot Y' + 2X \cdot Y) = 2 + 2Y' \rightarrow$

$X^2 e^{X^2Y} \cdot Y' + 2Xe^{X^2Y} \cdot Y = 2 + 2Y' \rightarrow$

$X^2 e^{X^2Y} \cdot Y' - 2Y' = 2 - 2XYe^{X^2Y} \rightarrow$

$Y'(X^2e^{X^2Y} - 2) = 2 - 2XYe^{X^2Y} \rightarrow$

$$Y' = \frac{2 - 2XYe^{X^2Y}}{X^2e^{X^2Y} - 2}$$

21.) $x^2 + y^2 = 1 \xrightarrow{D}$

$2x + 2yy' = 0 \rightarrow 2yy' = -2x \rightarrow$

$y' = \frac{-2x}{2y} \longrightarrow \boxed{y' = \frac{-x}{y}}$; $\xrightarrow{D}$

$$y'' = \frac{y \cdot (-1) - (-x) \cdot y'}{y^2}$$

$$= \frac{-y + x \cdot y'}{y^2} = \frac{-y + x \cdot \left(\frac{-x}{y}\right)}{y^2} \cdot \frac{y}{y}$$

$$= \frac{-y^2 - x^2}{y^3} = \frac{-(x^2 + y^2)}{y^3} = \frac{-1}{y^3} \longrightarrow$$

$$\boxed{y'' = \frac{-1}{y^3}}$$

26.) $xy + y^2 = 1 \xrightarrow{D}$

$xy' + (1)y + 2yy' = 0 \longrightarrow$

$y'(x + 2y) = -y \longrightarrow \boxed{y' = \frac{-y}{x + 2y}}$; $\xrightarrow{D}$

$$y'' = \frac{(x+2y) \cdot -y' - (-y) \cdot (1 + 2y')}{(x+2y)^2}$$

$$= \frac{-(x+2y) \cdot \frac{-y}{x+2y} + y + 2y \cdot \frac{-y}{x+2y}}{(x+2y)^2} \cdot \frac{x+2y}{x+2y}$$

$$= \frac{(x+2y) \cdot y + y \cdot (x+2y) - 2y^2}{(x+2y)^3}$$

$$= \frac{xy + 2y^2 + xy + \cancel{2y^2} - \cancel{2y^2}}{(x+2y)3}$$

$$= \frac{2xy + 2y^2}{(x+2y)^3} \longrightarrow \boxed{y'' = \frac{2xy + 2y^2}{(x+2y)^3}}$$

28.) $XY + Y^2 = 1 \xrightarrow{D} XY' + (1)Y + 2YY' = 0 \rightarrow$

$Y'(X+2Y) = -Y \rightarrow \boxed{Y' = \dfrac{-Y}{X+2Y}}$ and

$X=0,\ Y=-1 \rightarrow \quad Y' = \dfrac{-(-1)}{0+2(-1)} = \dfrac{-1}{2}$;

$\xrightarrow{D} Y'' = \dfrac{(X+2Y)(-Y') - (-Y)(1+2Y')}{(X+2Y)^2}$ and

$X=0,\ Y=-1,\ Y' = -1/2 \rightarrow$

$Y'' = \dfrac{(-2)(1/2) + (-1)(0)}{(-2)^2} = \dfrac{-1}{4}$

30.) $(X^2+Y^2)^2 = (X-Y)^2 \xrightarrow{D}$

$2(X^2+Y^2)\cdot(2X+2YY') = 2(X-Y)(1-Y') \rightarrow$

$4X(X^2+Y^2) + 4YY'(X^2+Y^2)$

$= 2(X-Y) - 2(X-Y)Y' \rightarrow$

$4YY'(X^2+Y^2) + 2(X-Y)Y'$

$= 2(X-Y) - 4X(X^2+Y^2) \rightarrow$

$Y'[4Y(X^2+Y^2) + 2(X-Y)]$

$= 2(X-Y) - 4X(X^2+Y^2) \rightarrow$

$Y' = \dfrac{2(X-Y) - 4X(X^2+Y^2)}{4Y(X^2+Y^2) + 2(X-Y)}$;

$X=1, Y=0 \rightarrow$ SLOPE $Y' = \dfrac{2-4}{2} = -1$,

$X=1, Y=-1 \rightarrow$ SLOPE $Y' = \dfrac{4-8}{-8+4} = \dfrac{-4}{-4} = -1$

31.) $x^2 + xy - y^2 = 1 \xrightarrow{D}$

$2x + xy' + (1)y - 2yy' = 0 \rightarrow$

$(x - 2y)y' = -2x - y \rightarrow$

$y' = \dfrac{-2x - y}{x - 2y} = \dfrac{2x + y}{2y - x}$

a.) $x = 2,\ y = 3 \rightarrow m = y' = \dfrac{4+3}{6-2} = \dfrac{7}{4}$ so

$y - 3 = \frac{7}{4}(x - 2)$ or $y = \frac{7}{4}x - \frac{1}{2}$

b.) $x = 2,\ y = 3 \rightarrow m = \dfrac{-4}{7}$ so

$y - 3 = \frac{-4}{7}(x - 2)$ or $y = \frac{-4}{7}x + \frac{29}{7}$

33.) $x^2 y^2 = 9 \xrightarrow{D}$

$x^2 \cdot 2yy' + (2x) \cdot y^2 = 0 \rightarrow$

$2x^2 y y' = -2xy^2 \rightarrow$

$y' = \dfrac{-2xy^2}{2x^2 y} \rightarrow y' = -\dfrac{y}{x}$

a.) $x = -1,\ y = 3 \rightarrow m = y' = \dfrac{-3}{-1} = 3$ so

$y - 3 = 3(x - (-1))$ or $y = 3x + 6$

b.) $x = -1,\ y = 3 \rightarrow m = \dfrac{-1}{3}$ so

$y - 3 = \frac{-1}{3}(x - (-1))$ or $y = \frac{-1}{3}x + \frac{8}{3}$

38.) $x \sin 2y = y \cos 2x \xrightarrow{D}$

$$x \cdot \cos 2Y \cdot 2Y' + (1) \cdot \sin 2Y$$
$$= Y \cdot -\sin 2X \cdot 2 + Y' \cdot \cos 2X \rightarrow$$

$$2X \cos 2Y \cdot Y' - \cos 2X \cdot Y'$$
$$= -2Y \sin 2X - \sin 2Y \rightarrow$$

$$Y'(2X \cos 2Y - \cos 2X) = -2Y \sin 2X - \sin 2Y \rightarrow$$

$$Y' = \frac{-2Y \sin 2X - \sin 2Y}{2X \cos 2Y - \cos 2X} = \frac{2Y \sin 2X + \sin 2Y}{\cos 2X - 2X \cos 2Y}$$

a.) $X = \frac{\pi}{4}$, $Y = \frac{\pi}{2} \rightarrow$

$$m = Y' = \frac{\pi \sin \frac{\pi}{2} + \sin \pi}{\cos \frac{\pi}{2} - \frac{\pi}{2} \cos \pi} = \frac{\pi(1) + 0}{0 - \frac{\pi}{2}(-1)} = \frac{\pi}{\frac{\pi}{2}} = 2$$

so $Y - \frac{\pi}{2} = 2\left(X - \frac{\pi}{4}\right)$ or $Y = 2X$

b.) $X = \frac{\pi}{4}$, $Y = \frac{\pi}{2} \rightarrow$

$m = \frac{-1}{2}$ so $Y - \frac{\pi}{2} = \frac{-1}{2}\left(X - \frac{\pi}{4}\right)$ or

$Y = -\frac{1}{2}X + \frac{5\pi}{8}$

39.) $Y = 2\sin(\pi X - Y)$, if $X = 1$, $Y = 0$ then
$0 = 2\sin(\pi) = 2(0) = 0$; $\xrightarrow{D}$
$Y' = 2\cos(\pi X - Y)\cdot(\pi - Y')$
$= 2\pi\cos(\pi X - Y) - 2\cos(\pi X - Y)\cdot Y' \rightarrow$
$Y' + 2\cos(\pi X - Y)\cdot Y' = 2\pi\cos(\pi X - Y) \rightarrow$
$Y'(1 + 2\cos(\pi X - Y)) = 2\pi\cos(\pi X - Y) \rightarrow$

$$Y' = \frac{2\pi\cos(\pi X - Y)}{1 + 2\cos(\pi X - Y)} ;$$

a.) $X = 1$, $Y = 0$ so slope $Y' = \dfrac{2\pi\cos\pi}{1 + 2\cos\pi}$
$= \dfrac{2\pi(-1)}{1 + 2(-1)} = 2\pi$; tangent line is
$Y - 0 = 2\pi(X - 1)$ or $Y = 2\pi X - 2\pi$

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b.) $X = 1$, $Y = 0$ so $Y' = 2\pi$ and slope is $m = \dfrac{-1}{2\pi}$; $\perp$ line is
$Y - 0 = \dfrac{-1}{2\pi}(X - 1) \rightarrow Y = \dfrac{-1}{2\pi}X + \dfrac{1}{2\pi}$.

41.) $X^2 + XY + Y^2 = 7$; if $Y = 0$ (crosses X-axis) $\rightarrow X^2 = 7 \rightarrow X = \sqrt{7}$, $X = -\sqrt{7}$;
$\xrightarrow{D}$ $2X + XY' + (1)Y + 2YY' = 0 \rightarrow$
$Y'(X + 2Y) = -2X - Y \rightarrow Y' = \dfrac{-2X - Y}{X + 2Y}$;

slope at $(\sqrt{7}, 0)$ is $Y' = \dfrac{-2\sqrt{7}}{\sqrt{7}} = -2$;

slope at $(-\sqrt{7}, 0)$ is $Y' = \dfrac{2\sqrt{7}}{-\sqrt{7}} = -2$.

$\circledast$

42.) $\boxed{XY + 2X - Y = 0} \xrightarrow{D} XY' + (1)Y + 2 - Y' = 0 \rightarrow$

$(X-1)Y' = -Y - 2 \rightarrow Y' = \dfrac{-Y-2}{X-1} = \dfrac{Y+2}{1-X}$;

line $2X + Y = 0 \rightarrow Y = -2X$ has SLOPE $= \boxed{-2}$. So set $Y' = \boxed{\frac{1}{2}} \rightarrow$

$\dfrac{Y+2}{1-X} = \dfrac{1}{2} \rightarrow 2Y + 4 = 1 - X \rightarrow$

$\boxed{X = -2Y - 3}$; SUB in $\circledast$ $\rightarrow$

$(-2Y-3)Y + 2(-2Y-3) - Y = 0 \rightarrow$

$-2Y^2 - 3Y - 4Y - 6 - Y = 0 \rightarrow$

$2Y^2 + 8Y + 6 = 0 \rightarrow Y^2 + 4Y + 3 = 0 \rightarrow$

$(Y+3)(Y+1) = 0 \rightarrow \boxed{Y = -3, X = 3}$ or

$\boxed{Y = -1, X = -1}$; these two points have tangent slopes $= \frac{1}{2}$, so normal $(\perp)$ lines at these points will have slope $= \boxed{-2}$; then normal lines are

$Y - (-3) = -2(X - 3)$ or $Y = -2X + 3$

and

$Y - (-1) = -2(X - (-1))$ or $Y = -2X - 3$

44.) $2Y^2 - XY^2 = X^3 \xrightarrow{D}$

$4YY' - (X \cdot 2YY' + (1)Y^2) = 3X^2 \rightarrow$

$Y'(4Y - 2XY) = 3X^2 + Y^2 \rightarrow$

$Y' = \dfrac{3X^2 + Y^2}{4Y - 2XY}$ and $X=1, Y=1 \rightarrow$

$Y' = \dfrac{3+1}{4-2} = 2$; then tangent line is

$Y - 1 = 2(X-1)$ or $\boxed{Y = 2X - 1}$; normal

is $Y - 1 = \frac{-1}{2}(X-1)$ or $\boxed{Y = \frac{-1}{2}X + \frac{3}{2}}$

47.) $\boxed{X^2 + 2XY - 3Y^2 = 0} \xrightarrow{D}$

$2X + 2X \cdot Y' + 2 \cdot Y - 6YY' = 0 \rightarrow$

$Y'(2X - 6Y) = -2X - 2Y \rightarrow$

$Y' = \dfrac{-2X - 2Y}{2X - 6Y} = \dfrac{\cancel{2}(-X-Y)}{\cancel{2}(X - 3Y)} = \dfrac{X+Y}{3Y-1}$;

$X = 1, Y = 1 \rightarrow Y' = \frac{2}{2} = 1$ so slope

$m = -1$ and $\perp$ line is

$Y - 1 = -1(X-1) = -X + 1 \rightarrow \boxed{Y = -X + 2}$;

find pts. of intersection:

$X^2 + 2X(-X+2) - 3(-X+2)^2 = 0 \rightarrow$

$X^2 - 2X^2 + 4X - 3(X^2 - 4X + 4) = 0 \rightarrow$

$-X^2 + 4X - 3X^2 + 12X - 12 = 0 \rightarrow$

$-4X^2 + 16X - 12 = 0 \rightarrow$

$-4(X^2 - 4X + 3) = 0 \rightarrow$

$-4(X-3)(X-1) = 0 \rightarrow X = 1, Y = 1$

or $X = 3, Y = -1$

51.) a.) $\underline{x^2+y^2=4}$, $\underline{x^2=3y^2}$ $\rightarrow$

$3y^2+y^2=4 \rightarrow 4y^2=4 \rightarrow y^2=1 \rightarrow$

$y=1$, $x^2=3 \rightarrow$ $\boxed{x=\sqrt{3},\ y=1}$, $\boxed{x=-\sqrt{3},\ y=1}$;

$y=-1$, $x^2=3 \rightarrow$ $\boxed{x=\sqrt{3},\ y=-1}$, $\boxed{x=-\sqrt{3},\ y=-1}$;

$\underline{x^2+y^2=4} \xrightarrow{D} 2x+2yy'=0 \rightarrow 2yy'=-2x \rightarrow$

$y'=\frac{-2x}{2y} \rightarrow \boxed{y_1'=\frac{-x}{y}}$;

$x^2=3y^2 \xrightarrow{D} 2x=6yy' \rightarrow y'=\frac{2x}{6y} \rightarrow \boxed{y_2'=\frac{x}{3y}}$;

show the slopes for each point are negative reciprocals :

$x=\sqrt{3},\ y=1 :\quad y_1'=\frac{-\sqrt{3}}{1}$, $y_2'=\frac{\sqrt{3}}{3}=\sqrt{3}$

$x=-\sqrt{3},\ y=1 :\quad y_1'=\frac{\sqrt{3}}{1}$, $y_2'=\frac{-\sqrt{3}}{\sqrt{3}}=\frac{-1}{\sqrt{3}}$

$x=\sqrt{3},\ y=-1 :\quad y_1'=\frac{-\sqrt{3}}{-1}=\sqrt{3}$, $y_2'=\frac{\sqrt{3}}{-3}=\frac{-1}{\sqrt{3}}$

$x=-\sqrt{3},\ y=-1 :\quad y_1'=\frac{\sqrt{3}}{-1}$, $y_2'=\frac{-\sqrt{3}}{-3}=\frac{1}{\sqrt{3}}$

53.) $xy^3+x^2y=6 \xrightarrow{D}$ (y is function, x is variable)

$x\cdot 3y^2y'+(1)\cdot y^3+x^2\cdot y'+2x\cdot y=0 \rightarrow$

$y'(3xy^2+x^2)=-2xy-y^3 \rightarrow$

$y'=\boxed{\frac{dy}{dx}=\frac{-2xy-y^3}{3xy^2+x^2}}$;

$XY^3 + X^2 Y = 6 \xrightarrow{D}$ (X is function, Y is variable)

$X \cdot 3Y^2 + X' \cdot Y^3 + X^2 \cdot (1) + 2XX' \cdot Y = 0 \rightarrow$

$X'(Y^3 + 2XY) = -3XY^2 - X^2 \rightarrow$

$$X' = \boxed{\frac{dX}{dY} = \frac{-3XY^2 - X^2}{2XY + Y^3}} \quad ;$$

it follows that

$$\frac{dY}{dX} = \frac{1}{\frac{dX}{dY}} \quad .$$

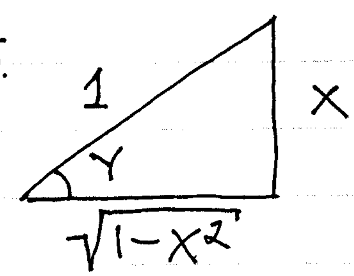
55.) $Y = \arcsin X \rightarrow X = \sin Y \xrightarrow{D}$

$= \frac{opp.}{hyp.}$

$1 = \cos Y \cdot Y' \rightarrow$

$$Y' = \frac{1}{\cos Y} = \frac{1}{\frac{adj.}{hyp.}}$$

$$= \frac{hyp.}{adj.} = \frac{1}{\sqrt{1-X^2}}$$



56.) a.) $Y = (\arcsin X)^2 \xrightarrow{D}$

$$Y' = 2(\arcsin X) \cdot \frac{1}{\sqrt{1-X^2}}$$

b.) $Y = \arcsin\left(\frac{1}{X}\right) \xrightarrow{D}$

$$y' = \frac{1}{\sqrt{1-(1/X)^2}} \cdot -X^{-2}$$