Section 4.7

Newton: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

2.) $x^3 + 3x + 1 = 0 \rightarrow f(x) = x^3 + 3x + 1 \xrightarrow{D}$

$f'(x) = 3x^2 + 3 \rightarrow$

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$$

$$= \frac{3x_n^3 + \cancel{3x_n} - x_n^3 - \cancel{3x_n} - 1}{3x_n^2 + 3} \rightarrow$$

$$\boxed{x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 3}}$$

$x_0 = 0 \rightarrow x_1 = \dfrac{2(0)^3 - 1}{3(0)^2 + 3} = \dfrac{-1}{3}$,

$x_1 = -1/3 = -0.3333\ldots \rightarrow x_2 = \dfrac{2\left(\frac{-1}{3}\right)^3 - 1}{3\left(\frac{-1}{3}\right)^2 + 3} = \dfrac{-29}{90}$,

$x_2 = \dfrac{-29}{90} \approx -0.3222 \rightarrow$

$x_3 = \dfrac{2\left(-\frac{29}{90}\right)^3 - 1}{3\left(-\frac{29}{90}\right)^2 + 3} \approx -0.3221$, so

solution $r \approx -0.3221$

3.) $x^4 + x - 3 = 0 \rightarrow f(x) = x^4 + x - 3 \xrightarrow{D}$

$f'(x) = 4x^3 + 1 \rightarrow$

$$x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1} = \frac{4x_n^4 + \cancel{x_n} - x_n^4 - \cancel{x_n} + 3}{4x_n^3 + 1}$$

$$\rightarrow \boxed{x_{n+1} = \frac{3x_n^4 + 3}{4x_n^3 + 1}}$$

$$x_0 = -1 \rightarrow x_1 = \frac{3(-1)^4+3}{4(-1)^3+1} = \frac{6}{-3} = -2,$$

$$x_1 = -2 \rightarrow x_2 = \frac{3(-2)^4+3}{4(-2)^3+1} = \frac{51}{-31} \approx -1.64516,$$

$$x_2 \approx -1.64516 \rightarrow x_3 = \frac{3x_2^4+3}{4x_2^3+1} \approx -1.48572,$$

$$x_3 \approx -1.48572 \rightarrow x_4 = \frac{3x_3^4+3}{4x_3^3+1} \approx -1.45381,$$

$$x_4 \approx -1.45381 \rightarrow x_5 = \frac{3x_4^4+3}{4x_4^3+1} \approx -1.45263,$$

$$x_5 \approx -1.45263 \rightarrow x_6 = \frac{3x_5^4+3}{4x_5^3+1} \approx -1.45263,$$

so solution 1 is $\boxed{r_1 \approx -1.452}$

$$x_0 = 1 \rightarrow x_1 = \frac{3(1)^4+3}{4(1)^3+1} = \frac{6}{5} = 1.2,$$

$$x_1 = 1.2 \rightarrow x_2 = \frac{3(1.2)^4+3}{4(1.2)^3+1} \approx 1.16542,$$

$$x_2 \approx 1.16542 \rightarrow x_3 = \frac{3x_2^4+3}{4x_2^3+1} \approx 1.16404,$$

$$x_3 \approx 1.16404 \rightarrow x_4 = \frac{3x_3^4+3}{4x_3^3+1} \approx 1.16403,$$

so solution 2 is $\boxed{r_2 \approx 1.164}$

5.) $x^4 - 2 = 0 \rightarrow f(x) = x^4 - 2 \xrightarrow{D} f'(x) = 4x^3 \rightarrow$

$$x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3} = \frac{4x_n^4 - x_n^4 + 2}{4x_n^3} \rightarrow$$

$$\boxed{x_{n+1} = \frac{3x_n^4 + 2}{4x_n^3}}$$

$$x_0 = 1 \rightarrow x_1 = \frac{3(1)^4 + 2}{4(1)^3} = \frac{5}{4} = 1.25,$$

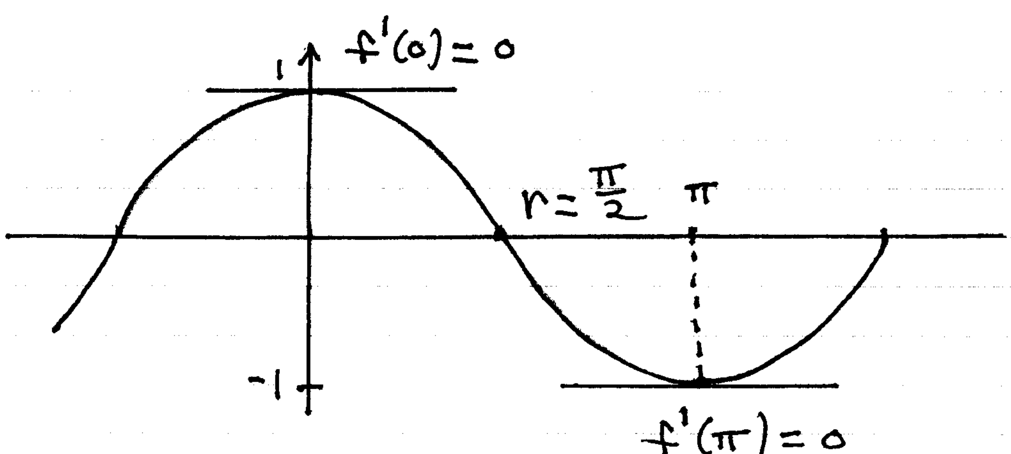
$$x_1 = 1.25 \rightarrow x_2 = \frac{3(1.25)^4 + 2}{4(1.25)^3} = 1.1935,$$

$$x_2 = 1.1935 \rightarrow x_3 = \frac{3(1.1935)^4 + 2}{4(1.1935)^3} \approx 1.18923$$

$$x_3 \approx 1.18923 \rightarrow x_4 = \frac{3x_3^4 + 2}{4x_3^3} \approx 1.18921,$$

so $2^{1/4} \approx 1.189$

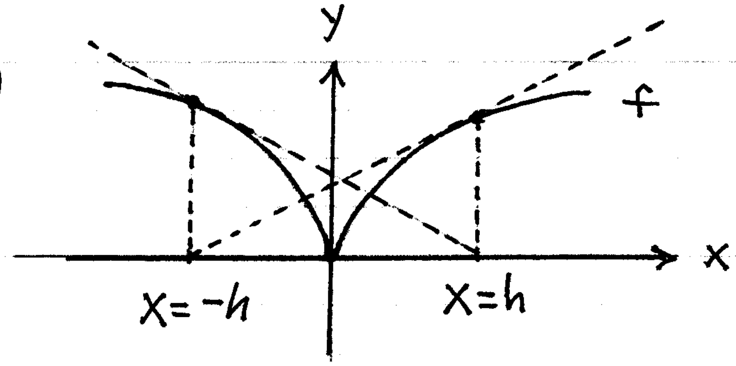
8.)



Since $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $x_0 \neq 0, \pm\pi, \pm 2\pi, \ldots$

since $f'(x_0) = 0$!

9.)



$$f(x)=\begin{cases}\sqrt{x}, & x\ge 0\\ -\sqrt{-x}, & x<0,\end{cases}$$

$$\xrightarrow{D} f'(x)=\begin{cases}\dfrac{1}{2\sqrt{x}}, & x\ge 0\\ \dfrac{-1}{2\sqrt{-x}}, & x<0\end{cases};$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$; if $x_0=h$, then

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=h-\frac{f(h)}{f'(h)}$$

$$=h-\frac{\sqrt{h}}{\frac{1}{2\sqrt{h}}}=h-2h=-h \rightarrow x_1=-h \rightarrow$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=-h-\frac{f(-h)}{f'(-h)}$$

$$=-h-\frac{-\sqrt{-(-h)}}{\frac{-1}{2\sqrt{-(-h)}}}=-h+2h=h \rightarrow x_2=h \rightarrow$$

$x_3=-h$, $x_4=h$, $x_5=-h, \cdots,$

so Newton's Method fails to find solution r.

10.) $x^{1/3} = 0 \rightarrow f(x) = x^{1/3} \xrightarrow{D} f'(x) = \dfrac{1}{3x^{2/3}} \rightarrow$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3x_n^{2/3}}} = x_n - 3x_n \rightarrow$$

$\boxed{x_{n+1} = -2x_n}$;

$x_0 = 1 \rightarrow x_1 = -2x_0 = -2(1) = -2$,

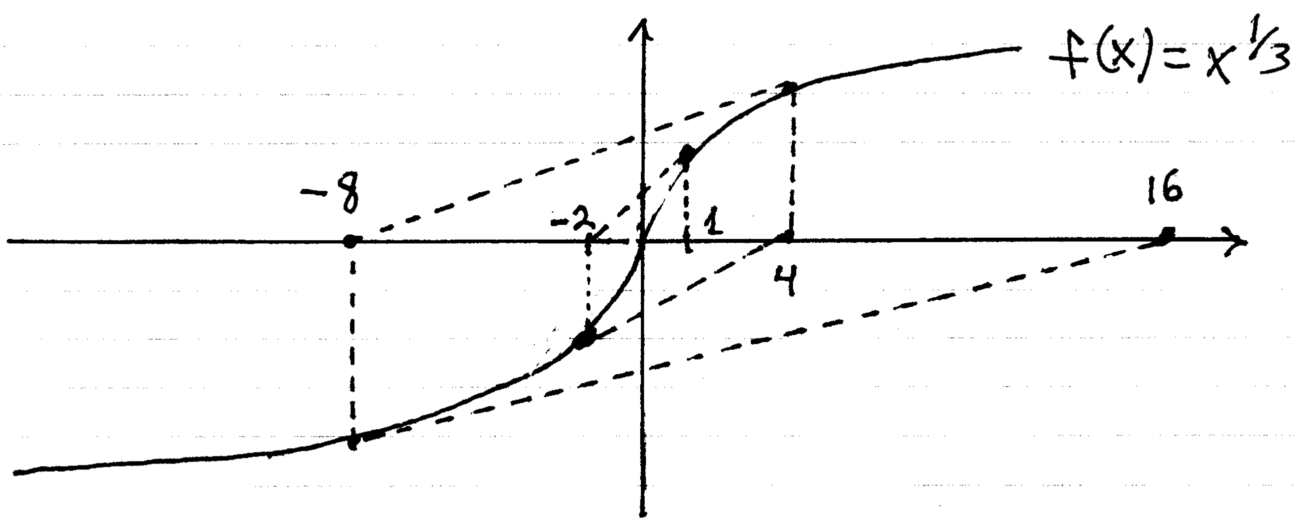
$x_1 = -2 \rightarrow x_2 = -2x_1 = -2(-2) = 4$,

$x_2 = 4 \rightarrow x_3 = -2x_2 = -2(4) = -8$,

$x_3 = -8 \rightarrow x_4 = -2x_3 = -2(-8) = 16$,

$x_4 = 16 \rightarrow \cdots$

$x_n = (-1)^n 2^n$, then $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} 2^n = \infty$.



13) $\tan x = 2x$ on $0 < x < \frac{\pi}{2} \rightarrow$

$\tan - 2x = 0 \rightarrow f(x) = \tan x - 2x$

$\xrightarrow{D} f'(x) = \sec^2 x - 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan x_n - 2x_n}{\sec^2 x_n - 2}$$

$$= \frac{x_n \sec^2 x_n - \cancel{2x_n} - \tan x_n + \cancel{2x_n}}{\sec^2 x_n - 2}$$

$$= \frac{x_n \cdot \frac{1}{\cos^2 x_n} - \frac{\sin x_n}{\cos x_n}}{\frac{1}{\cos^2 x_n} - 2} \cdot \frac{\cos^2 x_n}{\cos^2 x_n}$$

$$\boxed{x_{n+1} = \frac{x_n - \sin x_n \cos x_n}{1 - 2\cos^2 x_n}}\ ;$$

$$x_0 = 1 \to x_1 = \frac{1 - \sin 1 \cdot \cos 1}{1 - 2\cos^2 1} \approx 1.31048,$$

$$x_1 \approx 1.31048 \to x_2 = \frac{x_1 - \sin x_1 \cos x_1}{1 - 2\cos^2 x_1} \approx 1.22393,$$

$$x_2 \approx 1.22393 \to x_3 = \frac{x_2 - \sin x_2 \cos x_2}{1 - 2\cos^2 x_2} \approx 1.17605,$$

$$x_3 \approx 1.17605 \to x_4 = \frac{x_3 - \sin x_3 \cos x_3}{1 - 2\cos^2 x_3} \approx 1.16593,$$

$$x_4 \approx 1.16593 \to x_5 = \frac{x_4 - \sin x_4 \cos x_4}{1 - 2\cos^2 x_4} \approx 1.16556,$$

so solution $r \approx 1.165$

18.) $\tan x = 0 \rightarrow f(x) = \tan x \xrightarrow{D}$

$f'(x) = \sec^2 x \rightarrow$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\tan x_n}{\sec^2 x_n} = x_n - \frac{\frac{\sin x_n}{\cos x_n}}{\frac{1}{\cos^2 x_n}}$$

$$= x_n - \frac{\sin x_n}{\cos x_n} \cdot \frac{\cos^2 x_n}{1} \rightarrow$$

$$\boxed{x_{n+1} = x_n - \sin x_n \cos x_n}$$

$x_0 = 3 \rightarrow x_1 = 3 - \sin 3 \cos 3 \approx 3.139707749$

$\rightarrow x_2 = x_1 - \sin x_1 \cos x_1 \approx 3.141592649$

$\rightarrow x_3 = x_2 - \sin x_2 \cos x_2 \approx 3.141592654$

$\rightarrow x_4 = x_3 - \sin x_3 \cos x_3 \approx 3.141592654$

21.) $y = x^3 + x^2,\ y = \frac{1}{x} \rightarrow x^3 + x^2 = \frac{1}{x}$

$\rightarrow x^4 + x^3 = 1 \rightarrow \underbrace{x^4 + x^3 - 1}_{f(x)} = 0$

$f'(x) = 4x^3 + 3x^2 \rightarrow$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + x_n^3 - 1}{4x_n^3 + 3x_n^2}$$

$$= \frac{4x_n^4 + 3x_n^3 - x_n^4 - x_n^3 + 1}{4x_n^3 + 3x_n^2} \rightarrow$$

$$\boxed{x_{n+1} = \frac{3x_n^4 + 2x_n^3 + 1}{4x_n^3 + 3x_n^2}} \; ;$$

$$x_0 = 1 \rightarrow x_1 = \frac{3(1)^4 + 2(1)^3 + 1}{4(1)^3 + 3(1)^2} = \frac{6}{7} \approx 0.85714 \, ,$$

$$x_2 = \frac{3x_1^4 + 2x_1^3 + 1}{4x_1^3 + 3x_1^2} \approx 0.82125$$

$$x_3 = \frac{3x_2^4 + 2x_2^3 + 1}{4x_2^3 + 3x_2^2} \approx 0.81918$$

$$x_4 = \frac{3x_3^4 + 2x_3^3 + 1}{4x_3^3 + 3x_3^2} \approx 0.81917 \, , \text{ so}$$

solution $r \approx 0.819$

29.) $f(x) = (x-1)^{40} = 0 \xrightarrow{D}$

$f'(x) = 40(x-1)^{39} \longrightarrow$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n-1)^{40}}{40(x_n-1)^{39}}$$

$$= x_n - \frac{1}{40}(x_n - 1) = x_n - \frac{1}{40}x_n + \frac{1}{40} \rightarrow$$

$$\boxed{x_{n+1} = \frac{39}{40}x_n + \frac{1}{40}} \; ;$$

$$x_0 = 2 \rightarrow x_1 = \frac{39}{40}(2) + \frac{1}{40} = \frac{79}{40} = 1.975 \, ,$$

$$x_2 = \frac{39}{40}x_1 + \frac{1}{40} = 1.950625 \, ,$$

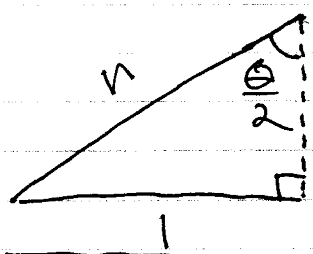
$$x_3 = \frac{39}{40}x_2 + \frac{1}{40} \approx 1.926859375$$

(Newton's Method goes slowly

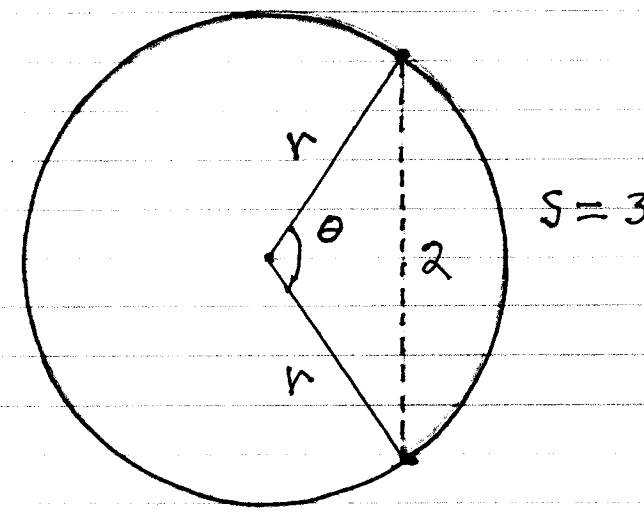
to the solution $r = 1$!)

30.) $0 < \theta < \pi$,

$r\theta = 3 \rightarrow \boxed{r = \frac{3}{\theta}}$,



and



$$\boxed{\sin\left(\frac{\theta}{2}\right) = \frac{1}{r}} = \frac{1}{\frac{3}{\theta}} = \frac{\theta}{3} \rightarrow$$

$$\frac{\sin\left(\frac{\theta}{2}\right)}{\theta} = \frac{1}{3} \rightarrow \underbrace{\frac{\sin\left(\frac{\theta}{2}\right)}{\theta} - \frac{1}{3}}_{f(\theta)} = 0 \ ;$$

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} \ ,$$

$$f'(\theta) = \frac{\theta \cdot \cos\left(\frac{\theta}{2}\right) \cdot \frac{1}{2} - \sin\left(\frac{\theta}{2}\right) \cdot (1)}{\theta^2} \longrightarrow$$

$$\theta_{n+1} = \theta_n - \frac{\frac{\sin\left(\frac{\theta_n}{2}\right)}{\theta_n} - \frac{1}{3}}{\frac{\frac{\theta_n}{2}\cos\left(\frac{\theta_n}{2}\right) - \sin\left(\frac{\theta_n}{2}\right)}{\cancel{\theta_n^2}}} \cdot \frac{6\theta_n^2}{6\cancel{\theta_n^2}}$$

$$= \theta_n - \frac{6\theta_n \sin\left(\frac{\theta_n}{2}\right) - 2\theta_n^2}{3\theta_n \cos\left(\frac{\theta_n}{2}\right) - 6\sin\left(\frac{\theta_n}{2}\right)}$$

$$= \frac{3\theta_n^2 \cos\left(\frac{\theta_n}{2}\right) - 6\theta_n \sin\left(\frac{\theta_n}{2}\right) - 6\theta_n \sin\left(\frac{\theta_n}{2}\right) + 2\theta_n^2}{3\theta_n \cos\left(\frac{\theta_n}{2}\right) - 6\sin\left(\frac{\theta_n}{2}\right)} \longrightarrow$$

$$\boxed{\theta_{n+1} = \frac{3\theta_n^2 \cos\left(\frac{\theta_n}{2}\right) - 12\theta_n \sin\left(\frac{\theta_n}{2}\right) + 2\theta_n^2}{3\theta_n \cos\left(\frac{\theta_n}{2}\right) - 6\sin\left(\frac{\theta_n}{2}\right)}};$$

$\theta_0 = 2 \longrightarrow$

$\theta_1 \approx 3.16084 \longrightarrow$

$\theta_2 \approx 2.99376 \longrightarrow$

$\theta_3 \approx 2.99156 \longrightarrow$

$\theta_4 \approx 2.99156 \longrightarrow \boxed{\theta \approx 2.9915}$

and $r = \frac{3}{\theta} \longrightarrow \boxed{r \approx 1.0028}$