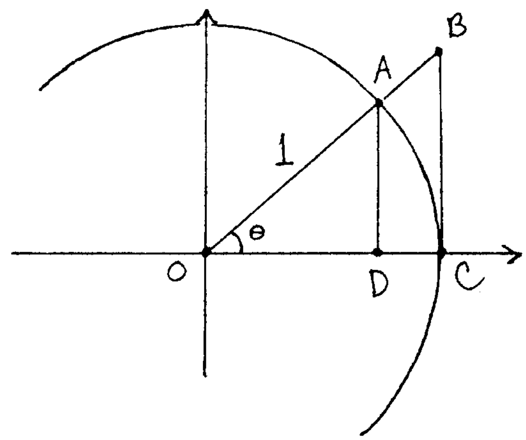
Math 21A
Koubа
A Useful Limit

Problem : Determine $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$, where θ is given in radians.



Start by assuming that $\theta > 0$ is in the first quadrant, and assume the circle has radius 1.

It follows that

$$\text{Area } \triangle OAD < \text{Area } \triangleleft OAC < \text{Area } \triangle OBC \longrightarrow$$

$$\frac{1}{2}(\cos\theta)(\sin\theta) < \frac{\theta}{2\pi} \cdot \pi(1)^2 < \frac{1}{2}(1)(\tan\theta) \longrightarrow$$

$$(\cos\theta)(\sin\theta) < \theta < \frac{\sin\theta}{\cos\theta} \longrightarrow$$

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \longrightarrow$$

$$\frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta \longrightarrow$$

$$\lim_{\theta \to 0} \frac{1}{\cos\theta} \geq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \geq \lim_{\theta \to 0} \cos\theta \longrightarrow$$

$$\frac{1}{1} \geq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \geq 1 \longrightarrow$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$ (similarly for $\theta < 0$) .