Math 21A (Fall 2017)
Kouba
Exam 1

Please PRINT your name here : KEY

Your Exam ID Number ___________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER PERSON TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 7 pages, including the cover page.

6. You may NOT use L'Hopital's Rule on this exam.

7. You may NOT use the shortcut for finding limits to infinity.

8. Using only a calculator to determine limits will receive little or no credit.

9. You will be graded on proper use of limit notation.

10. You have until 3 p.m. sharp to finish the exam.

1.) (7 pts. each) Determine the following limits.

a.) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2+x-6} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 2} \dfrac{\cancel{x-2}}{(\cancel{x-2})(x+3)} = \dfrac{1}{5}$

b.) $\lim\limits_{x \to 0} \dfrac{\frac{1}{x+1} - 1}{\frac{x}{1}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 0} \left(\dfrac{1}{x+1} - \dfrac{x+1}{x+1}\right) \cdot \dfrac{1}{x}$

$= \lim\limits_{x \to 0} \dfrac{\cancel{1} - x - \cancel{1}}{x+1} \cdot \dfrac{1}{x} = \lim\limits_{x \to 0} \dfrac{-1}{x+1} = -1$

c.) $\lim\limits_{h \to \infty} \dfrac{x^{-2}}{x^{-3}+x^{-1}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to \infty} \dfrac{\frac{1}{x^2}}{\frac{1}{x^3}+\frac{1}{x}} \cdot \dfrac{x}{x}$

$= \lim\limits_{x \to \infty} \dfrac{1/x}{\frac{1}{x^2}+1} = \dfrac{0}{0+1} = 0$

d.) $\lim\limits_{x \to +3^-} \dfrac{x^2-5}{3-x} = \dfrac{4}{0^+} = +\infty$

2.9

$x = 3$

e.) $\lim\limits_{x \to 1} \dfrac{2-\sqrt{x+3}}{x-1} \cdot \dfrac{2+\sqrt{x+3}}{2+\sqrt{x+3}}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 1} \dfrac{4-(x+3)}{(x-1)(2+\sqrt{x+3})} = \lim\limits_{x \to 1} \dfrac{-(\cancel{x-1})}{(\cancel{x-1})(2+\sqrt{x+3})}$

$= \dfrac{-1}{2+2} = \dfrac{-1}{4}$

f.) $\lim_{x \to 0} x^2(\sin x + \sin(1/x))^3$ (HINT: Use the Squeeze Principle.)

$-1 \le \sin x \le 1$ and $-1 \le \sin\left(\frac{1}{x}\right) \le 1 \to$
$-2 \le \sin x + \sin\frac{1}{x} \le 2 \to -8 \le \left(\sin x + \sin\frac{1}{x}\right)^3 \le 8 \to$
$-8x^2 \le x^2\left(\sin x + \sin\frac{1}{x}\right)^3 \le 8x^2$; and
$\lim_{x \to 0} -8x^2 = 0 = \lim_{x \to 0} 8x^2$ so $\lim_{x \to 0} x^2\left(\sin x + \sin\frac{1}{x}\right)^3 = 0$
by the Squeeze Principle.

2.) a.) (6 pts.) Determine the domain for $f(x) = \frac{3}{4 - \sqrt{x-5}}$. We need
$x - 5 \ge 0 \to x \ge 5$; and $4 - \sqrt{x-5} \ne 0 \to$
$4 \ne \sqrt{x-5} \to 16 \ne x - 5 \to x \ne 21$;
Domain: all $x \ge 5$ but $x \ne 21$

b.) (6 pts.) Determine the range for $f(x) = 3 - 5\cos x$.

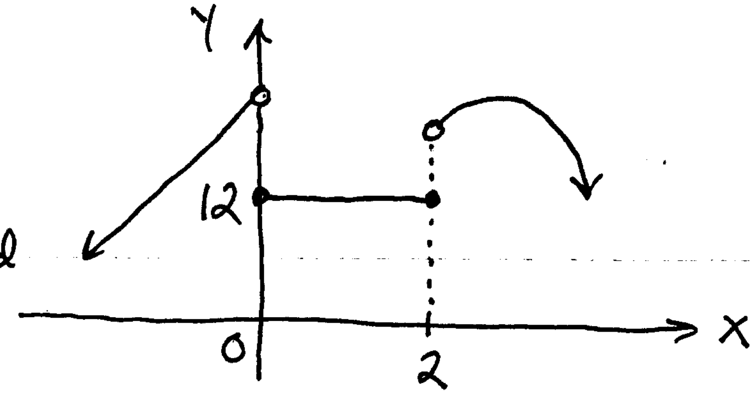
$-1 \le \cos x \le 1 \to -5 \le -5\cos x \le 5 \to$
$-2 \le \underbrace{3 - 5\cos x}_{y} \le 8$, so range is
$-2 \le y \le 8$

3.) (8 pts.) For what x-values is the following function continuous ? Briefly explain.

$y = \frac{x \ln x}{x^2 - 9}$; $y = x$ and $y = x^2 - 9$ are cont.
for all x-values (polynomials) ; $y = \ln x$
is cont. for $x > 0$ (well known) ; then
$y = x \ln x$ is cont. for all $x > 0$ (product)
and $y = \frac{x \ln x}{x^2 - 9}$ is cont. for all
$x > 0$ EXCEPT where $x^2 - 9 = (x-3)(x+3) = 0$,
i.e. EXCEPT $x = 3$ and ~~$x = -3$~~ .

4.) (8 pts.) Use limits and a fake graph to determine the value(s) of constants A and B so that the following function is continuous for all values of x :

$$f(x) = \begin{cases} Ax + B, & \text{if } x < 0 \\ 12 \ , & \text{if } 0 \le x \le 2 \\ Bx^2 - A, & \text{if } x > 2 \ . \end{cases}$$



We need

$$\begin{cases} \lim_{x \to 0^-} (AX+B) = 12 \text{ and} \\ \lim_{x \to 2^+} (BX^2 - A) = 12 \end{cases} \rightarrow$$

$\boxed{B = 12}$ and $B(4) - A = 12 \rightarrow$

$(12)(4) - A = 12 \rightarrow A = 48 - 12 \rightarrow$

$\boxed{A = 36}$

5.) (8 pts.) Use the Intermediate Value Theorem to prove that the equation $x^3 = x^2 + 5$ is solvable. This is a writing exercise.

$x^3 - x^2 - 5 = 0$ so let $\underline{f(x) = x^3 - x^2 - 5}$ and $\underline{m = 0}$; f is <u>cont.</u> for all x (polynomial) ; by trial and error $f(0) = -5$ and $f(3) = 27 - 9 - 5 = 13$; $m = 0$ is <u>between</u> $f(0)$ and $f(3)$, so choose <u>interval</u> $[0, 3]$. By IMVT there is at least one x-value c, $0 \le c \le 3$, so that $f(c) = m$, i.e., $c^3 - c^2 - 5 = 0$ and the equation is solvable.

6.) (8 pts.) Give an ε, δ-proof for the following limit. This is a writing exercise.
$\lim_{x \to 2} (x^2 + x) = 6$

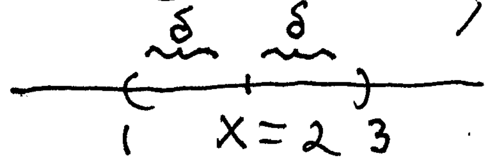
Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x-2| < \delta$, then $|f(x) - 6| < \varepsilon$. Begin with $|f(x)-6| < \varepsilon$ and "solve for" $|x-2|$. Then

$|f(x)-6| < \varepsilon$ iff $|(x^2+x)-6| < \varepsilon$

iff $|(x+3)(x-2) < \varepsilon$

iff $|x+3||x-2| < \varepsilon$.

We need to "eliminate" $|x-3|$, so assume $\delta \le 1 \rightarrow$



$1 < x < 3 \longrightarrow$

$4 < |x+3| < 6$. Then (Pinch Step)

$|x+3||x-2| < 6|x-2| < \varepsilon$

iff $|x-2| < \frac{1}{6}\varepsilon$. Now choose $\delta = \min\{1, \frac{1}{6}\varepsilon\}$. This completes the proof.

Q.E.D.

7.) (8 pts.) Use limits to find all horizontal asymptotes for $y = \dfrac{2^x + 3^x}{2^x + 5^x}$.

$$\lim_{x\to\infty} \frac{2^x+3^x}{2^x+5^x} \cdot \frac{1/5^x}{1/5^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x}{\left(\frac{2}{5}\right)^x + 1} = \frac{0+0}{0+1} = 0$$

so H.A. is $\boxed{y=0}$;

$$\lim_{x\to-\infty} \frac{2^x+3^x}{2^x+5^x} \cdot \frac{1/2^x}{1/2^x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to-\infty} \frac{1+\left(\frac{3}{2}\right)^x}{1+\left(\frac{5}{2}\right)^x}$$

$$= \frac{1+\left(3/2\right)^{-\infty}}{1+\left(5/2\right)^{-\infty}} = \frac{1+\left(\frac{2}{3}\right)^{\infty}}{1+\left(\frac{2}{5}\right)^{\infty}} = \frac{1+0}{1+0} = 1$$

so H.A. is $\boxed{y=1}$

8.) (6 pts.) Find a tilted (slant) asymptote for the graph of $f(x) = \dfrac{3x^3 + x^2 - x}{x^2+1}$.

$$\begin{array}{r l}
 & 3x + 1 \\
x^2+1 \,\big)\!\! & \overline{3x^3 + x^2 - x} \\
 & -(3x^3 \quad + 3x) \\
\hline
 & \quad x^2 - 4x \\
 & \; -(x^2 + 1) \\
\hline
 & \quad -4x - 1
\end{array}$$

so $y = 3x+1$ is a T.A.

The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

1.) Determine the exact value of the following continued fraction.

$$1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{1+\cfrac{6}{1+\cdots}}}} = L \longrightarrow$$

$$1 + \frac{6}{L} = L \longrightarrow$$

$$L + 6 = L^2 \longrightarrow$$

$$0 = L^2 - L - 6 = (L-3)(L+2)$$

$$\rightarrow \boxed{L = 3} \text{ or } \cancel{L = -2}$$