PRACTICE EXAM 2 SOLUTIONS

Math 21A
Kouba
Exam 2

Please PRINT your name here : KEY

Your Exam ID Number ------------

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You will be graded on proper use of derivative notation.

7. You have until 9:40 a.m. sharp to finish the exam.

1.) (5 pts. each) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

a.) $y = \pi + (5x+1)^{-4}$

$$Y' = -4(5X+1)^{-5} \cdot 5$$

b.) $f(x) = \sec x \cdot \tan 3x$

$$f'(x) = \sec x \cdot \sec^2 3x \cdot 3 + \sec x \tan x \cdot \tan 3x$$

c.) $g(x) = \sin(\cos^3(x^4))$

$$g'(x) = \cos(\cos^3(x^4)) \cdot 3\cos^2(x^4) \cdot -\sin(x^4) \cdot 4x^3$$

d.) $y = x^5 + 8^{-x^2}$

$$Y' = 5x^4 + 8^{-x^2} \cdot (-2x) \cdot \ln 8$$

e.) $y = \dfrac{4 - \ln x}{10 + \log_2(3x+7)}$

$$Y' = \frac{(10 + \log_2(3X+7)) \cdot \frac{-1}{X} - (4 - \ln X) \cdot \frac{1}{3X+7} \cdot 3 \cdot \frac{1}{\ln 2}}{(10 + \log_2(3X+7))^2}$$

f.) $y = x^{\ln x}$

$$\ln Y = \ln x^{\ln x} = \ln x \cdot \ln x = (\ln x)^2 \xrightarrow{D}$$

$$\frac{1}{Y} Y' = 2(\ln x) \cdot \frac{1}{X} \rightarrow Y' = X^{\ln x} \cdot 2\frac{\ln x}{X}$$

2.) (7 pts.) Use $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to differentiate the function $f(x) = \dfrac{x+7}{3-x}$.

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{(x+h)+7}{3-(x+h)} - \dfrac{x+7}{3-x}}{h/1}$$

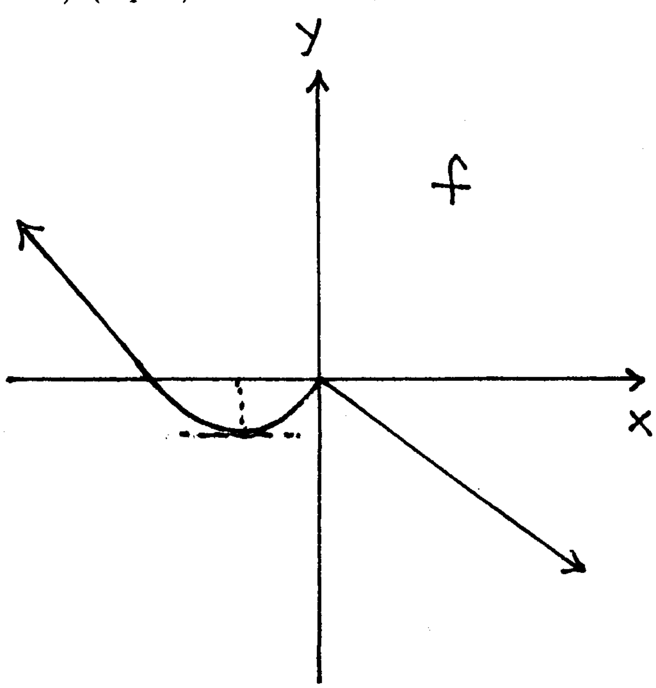
$$= \lim_{h\to 0} \frac{(x+h+7)(3-x)-(x+7)(3-x-h)}{(3-x-h)\,(3-x)} \cdot \frac{1}{h}$$

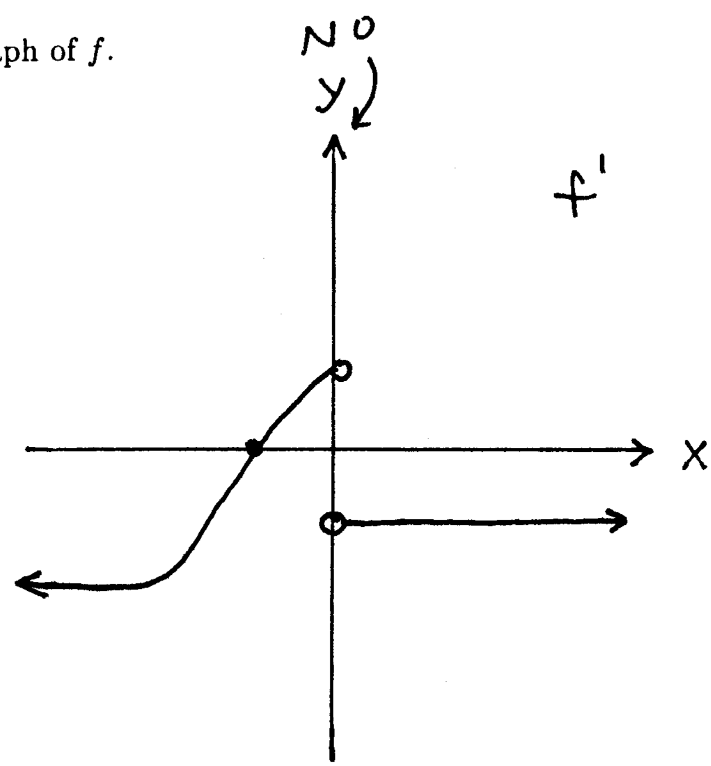
$$= \lim_{h\to 0} \frac{3x+3h+21-x^2-hx-7x-[3x-x^2-hx+21-7x-7h]}{(3-x-h)(3-x)\cdot h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h+\cancel{21}-\cancel{x^2}-\cancel{hx}-\cancel{7x}-\cancel{3x}+\cancel{x^2}+\cancel{hx}-\cancel{21}+\cancel{7x}+7h}{(3-x-h)(3-x)\cdot h}$$

$$= \lim_{h\to 0} \frac{10\cancel{h}}{(3-x-h)(3-x)\cancel{h}} = \frac{10}{(3-x)^2}$$

3.) (7 pts.) Sketch the graph of f' using the graph of f.





4.) (7 pts.) Assume that y is a function of x. Determine Y' and Y'' for the graph of $xy = y^2 - 1$ at $x = 0, y = 1$.

$$XY = Y^2 - 1 \xrightarrow{D} XY' + (1)Y = 2YY' \rightarrow$$

$$XY' - 2YY' = -Y \rightarrow Y'(X - 2Y) = -Y \rightarrow$$

$$\boxed{Y' = \frac{-Y}{X - 2Y}}$$ and $X = 0, Y = 1$ so

$$Y' = \frac{-1}{-2} = \frac{1}{2} \; ; \; Y'' = \frac{(X-2Y)(-Y') - (-Y)(1-2Y')}{(X-2Y)^2}$$

and $X = 0, Y = 1, Y' = 1/2$ so

$$Y'' = \frac{(-2)(-1/2) + (1)(0)}{(-2)^2} = \frac{1}{4}$$

5.) (5 pts. each) Let $f(x) = x^2 e^x$.

a.) Solve $f'(x) = 0$ for x and set up a sign chart for f'.

$$f'(x) = x^2 e^x + 2xe^x = xe^x(x+2) = 0$$

+	0	—	0	+	f'
	$X = -2$		$X = 0$		

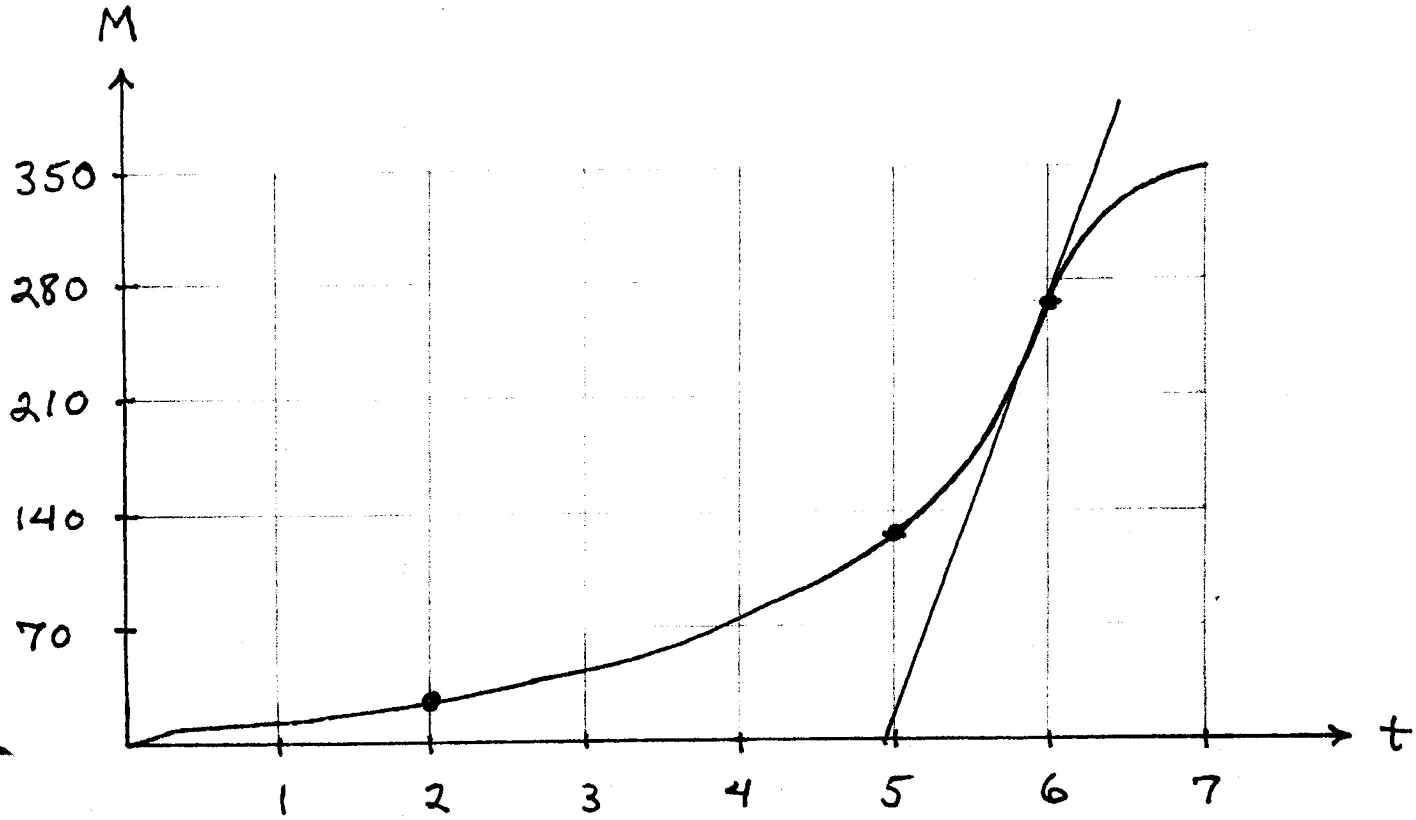
b.) Solve $f''(x) = 0$ for x and set up a sign chart for f''.

$$f''(x) = (1)e^x(x+2) + xe^x(x+2) + xe^x(1)$$
$$= e^x(x + 2 + x^2 + 2x + x) = e^x(x^2 + 4x + 2) = 0$$

$$\rightarrow X = \frac{-4 \pm \sqrt{16 - 8}}{2}$$
$$= \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

+	0	—	0	+	f''
	$X = -2 - \sqrt{2} \approx -3.4$		$X = -2 + \sqrt{2} \approx -0.6$		

6.) Assume that the given graph represents the total amount of money $M(t)$ in dollars ($) that you have spent at the UC Davis bookstore after t days.



a.) (3 pts.) How much money did you spend in the first 5 days ? $125

b.) (3 pts.) How much money did you spend after the 2nd day and through the entire 6th day ? $260 − $30 = $230

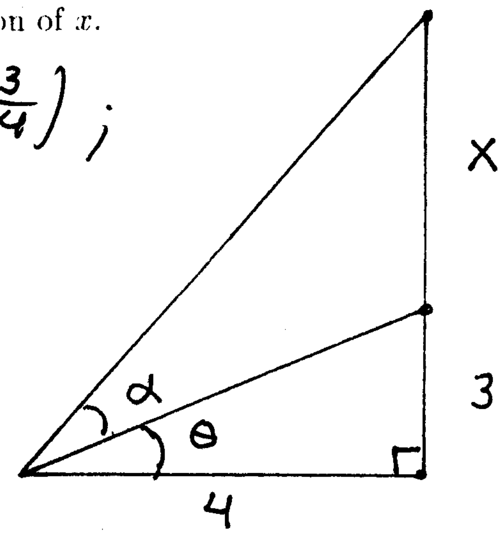
c.) (3 pts.) What was your average rate of spending ($ per day) for the first 7 days?

$$AVE = \frac{350-0}{7-0} = \$50 / day$$

d.) (3 pts.) What was your instantaneous rate of spending ($ per day) when $t = 6$ days ?

Slope at $t=6$: $\frac{260}{1} = \$260 / day$

7.) (6 pts.) Consider the given diagram. Write α as a function of x.



$$\tan\theta = \frac{3}{4} \rightarrow \theta = \arctan\left(\frac{3}{4}\right);$$

$$\tan(\alpha+\theta) = \frac{x+3}{4} \rightarrow$$

$$\alpha+\theta = \arctan\left(\frac{x+3}{4}\right) \rightarrow$$

$$\alpha = \arctan\left(\frac{x+3}{4}\right) - \theta \rightarrow$$

$$\alpha = \arctan\left(\frac{x+3}{4}\right) - \arctan(3/4)$$

8.) (6 pts.) Let $f(x) = x + 5\arctan(1/x)$. Solve $f'(x) = 0$ for x.

$$f'(x) = 1 + 5\cdot\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2}$$

$$= 1 + \frac{-5}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{5}{x^2+1}$$

$$= \frac{x^2-4}{x^2+1} = \frac{(x-2)(x+2)}{x^2+1} = 0 \rightarrow$$

$$(x-2)(x+2) = 0 \longrightarrow x = 2,\ x = -2$$

9.) (7 pts.) Differentiate the following function and SIMPLIFY your answer as much as possible : $f(x) = (x-3)\sqrt{6x-x^2} + 9\arcsin\left(\frac{x-3}{3}\right)$.

$$f'(x) = (x-3)\cdot\frac{1}{2}(6x-x^2)^{-1/2}\cdot(6-2x) + (1)\cdot\sqrt{6x-x^2} + 9\cdot\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\cdot\frac{1}{3}$$

$$= \frac{(x-3)(3-x)}{\sqrt{6x-x^2}} + \frac{6x-x^2}{\sqrt{6x-x^2}} + \frac{3}{\sqrt{\frac{9}{9}-\frac{x^2-6x+9}{9}}}$$

$$= \frac{3x-x^2-9+3x+6x-x^2}{\sqrt{6x-x^2}} + \frac{3}{\frac{\sqrt{6x-x^2}}{3}}$$

$$= \frac{12x-2x^2-9}{\sqrt{6x-x^2}} + \frac{9}{\sqrt{6x-x^2}} = \frac{2(6x-x^2)}{\sqrt{6x-x^2}}$$

$$= 2\sqrt{6x-x^2}$$

10.) (8 pts.) Consider the function $f(x) = ax^2 + bx + c$, where a, b, and c are unknown constants. Assume that the graph of f has a maximum value at $x = -1$ and $y = 2$ and assume $|a| = 3$. Solve for constants a, b, and c.

$f(x) = ax^2 + bx + c$ (parabola)

$\xrightarrow{D} f'(x) = 2ax + b$

and $f'(-1) = 2a(-1) + b = 0$

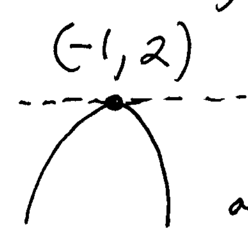
$\rightarrow \boxed{b = 2a}$; and

$f(-1) = a(-1)^2 + b(-1) + c = 2$

$\rightarrow \boxed{a - b + c = 2}$; then

$b = 2(-3) \rightarrow \boxed{b = -6}$ and

$(-3) - (-6) + c = 2 \rightarrow \boxed{c = -1}$



MAX $f'(-1) = 0$ and $a < 0$! so $|a| = 3$ and $\boxed{a = -3}$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) A beetle crawls along a thin rod on the x-axis from $x = 0$ in. to $x = 16$ in. at the rate of 3 in./min. The temperature of the rod at point x is $40 + 12\sqrt{x}$ degrees Fahrenheit (^{o} F). At what rate (^{o} F per min.) is the temperature of the rod under the beetle changing when the beetle is at $x = 9$ in. ?

$$\frac{dx}{dt} = 3 \text{ in./min}$$

$$T = 40 + 12\sqrt{x} \xrightarrow{D}$$

$$\frac{dT}{dt} = 12 \cdot \frac{1}{2} x^{-1/2} \cdot \frac{dx}{dt}$$

$$= \frac{6}{\sqrt{x}} \cdot \frac{dx}{dt}$$ and $x = 9$ in. $\longrightarrow$

$$\frac{dT}{dt} = \frac{6}{3} \cdot 3 = 6 \, {}^\circ F/\text{min}.$$