PRACTICE EXAM 3 SOLUTIONS

Math 21A
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ------------

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 9 pages, including the cover page.

6. You will be graded on proper use of derivative and limit notation.

7. You have until 9:40 a.m. sharp to finish the exam.

1.) (5 pts. each) Use L'Hopital's Rule to evaluate the following limits.

a.) $\lim_{x\to 0} \frac{\sin 6x}{\sin 3x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{\cos 6x \cdot 6}{\cos 3x \cdot 3} = \frac{(1)(6)}{(1)(3)} = 2$

b.) $\lim_{x\to\infty} \frac{x^2}{5x+e^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{2x}{5+e^x}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

c.) $\lim_{x\to 0+} x^2 \ln x = \text{"}0 \cdot \infty\text{"} = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0^+} -\frac{1}{2}x^2 = 0$

d.) $\lim_{x\to\infty} (3+2x)^{1/x} \overset{\text{"}\infty^0\text{"}}{=} \lim_{x\to\infty} e^{\ln (3+2x)^{1/x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(3+2x)}$

$= e^{\lim_{x\to\infty} \frac{\ln(3+2x)}{x}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{3+2x}\cdot 2}{1}} = e^0 = 1.$

2.) (7 pts.) Use a differential to show that $(8+\frac{h^2}{2})^{1/3} \approx 2+\frac{h^2}{24}$ for small h.

Let $f(x) = x^{1/3} \xrightarrow{D} f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$;

$x: 8 \to 8+\frac{1}{2}h^2$ so $\Delta x = \frac{1}{2}h^2$;

$\Delta f = f(8+\frac{1}{2}h^2) - f(8) = (8+\frac{1}{2}h^2)^{1/3} - 2$,

$df = f'(8)\cdot\Delta x = \frac{1}{3(8)^{2/3}}\cdot\frac{1}{2}h^2 = \frac{1}{24}h^2$, then

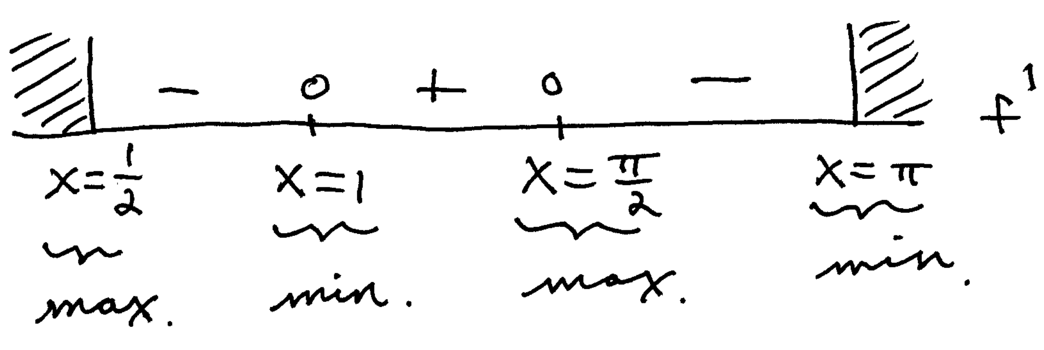
$\Delta f \approx df \to (8+\frac{1}{2}h^2)^{1/3} - 2 \approx \frac{1}{24}h^2 \to$

$(8+\frac{1}{2}h^2)^{1/3} \approx 2 + \frac{1}{24}h^2$

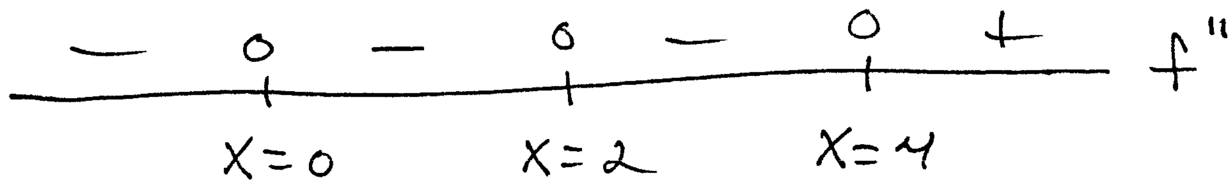
3.) a.) (7 pts.) Assume that $y = f(x)$ is defined on the interval $[\frac{1}{2}, \pi]$ with $f'(x) = \log x \cdot \cos x$. Determine all x-values corresponding to relative extrema for f.

$f'(x) = \log x \cdot \cos x = 0$ on $[0, \pi] \to$

$\log x = 0 \to x = 1$ or $\cos x = 0 \to x = \frac{\pi}{2}$;



4.) (7 pts.) Assume that the second derivative of function f is $f''(x) = x^4(x-2)^2(x-4)^5$. Determine all x-values corresponding to inflection points for f.



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Inflection Point

5.) a.) (2 pts.) State the Mean Value Theorem (MVT). assume f is cont. on $[a,b]$ and diff. on (a,b). Then there is at least one c, $a<c<b$, so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

b.) (5 pts.) Determine if the following function satisfies the assumptions of the MVT on the given closed interval. If so, find all values of c guaranteed by the conclusion of the MVT.

$f(x) = \sqrt{1-x}$ on $[0,1]$ $\quad$ $f(X) = \sqrt{1-X}$ is cont. on $[0,1]$ since it is the composition of cont. functions $Y=\sqrt{X}$ and $Y = 1-X$;

$f'(x) = \frac{1}{2}(1-x)^{-1/2} \cdot (-1) = \frac{-1}{2\sqrt{1-x}}$ so f is diff. on $(0,1)$. Then $f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{0-1}{1} = -1 \rightarrow$

$\frac{-1}{2\sqrt{1-c}} = -1 \rightarrow \frac{1}{2} = \sqrt{1-c} \rightarrow \frac{1}{4} = 1-c \rightarrow c = 3/4$

6.) (10 pts.) Consider the function $f(x) = x^2(x-3)$ on the closed interval $[-1, 4]$. Determine where f is increasing ($\uparrow$), decreasing ($\downarrow$), concave up ($\cup$), and concave down ($\cap$). Identify all relative and absolute extrema, inflection points, and x- and y-intercepts. Sketch the graph. The first and second derivatives are $f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$.

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0$

$+ \quad 0 \quad - \quad 0 \quad +$ $\quad f'$

$x=-1$	$x=0$	$x=2$	$x=4$
$y=-4$	$y=0$	$y=-4$	$y=16$
abs. min.	rel. max.	abs. min.	abs. max.

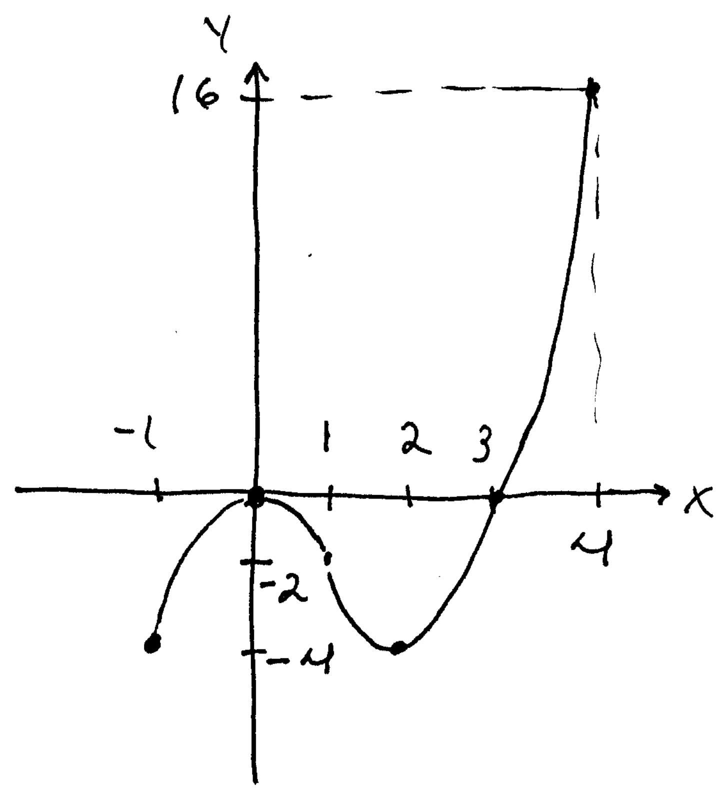
$f''(x) = 6x - 6 = 6(x-1) = 0$

$- \quad 0 \quad +$ $\quad f'$

$x=-1$ $\qquad$ $\left.\begin{matrix} x=1 \\ y=-2 \end{matrix}\right\}$ infl. ~~pt~~. $\qquad$ $x=4$

f is $\uparrow$ for $-1<x<0$, $2<x<4$,
f is $\downarrow$ for $0<x<2$,
f is $\cup$ for $1<x<4$,
f is $\cap$ for $-1<x<1$;

$x=0$: $y=0$
$y=0$: $x=0$, $x=3$



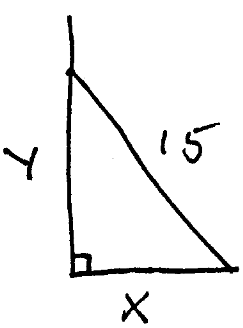
7.) (7 pts.) Determine the following limit: $\lim_{x \to 0^+} \sin x \cdot \ln x$ $= \text{"}0 \cdot -\infty\text{"}$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{\sin^2 x} \cdot \cos x}$$

$$= \lim_{x \to 0^+} -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x}$$

$$= -(1) \cdot \frac{\sin 0}{\cos 0} = -\frac{0}{1} = 0$$

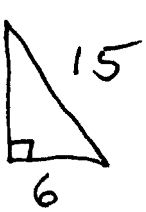
8.) (7 pts.) A 15-foot ladder is leaning against a wall. If the base of the ladder is pushed toward the wall at the rate of 2 ft./sec., at what rate is the top of the ladder moving up the wall when the base of the ladder is 6 ft. from the wall ?



assume $\frac{dx}{dt} = -2$ ft./sec.; find $\frac{dY}{dt}$ when $x = 6$ ft. :

$$x^2 + Y^2 = 15^2 \xrightarrow{D} 2x \cdot \frac{dx}{dt} + 2Y \frac{dY}{dt} = 0$$

$$(6)(-2) + \sqrt{189} \cdot \frac{dY}{dt} = 0 \rightarrow$$



$6^2 + Y^2 = 15^2$

$\rightarrow Y = \sqrt{189}$

≈ 13.7

$$\frac{dY}{dt} = \frac{12}{\sqrt{189}} \approx 0.87 \text{ ft./sec.}$$

9.) (7 pts.) Use differentials to estimate the value of $\sqrt{96}$.

Let $f(x) = \sqrt{x}$, $x : 100 \to 96$ so $\Delta x = -4$,

$f'(x) = \dfrac{1}{2\sqrt{x}}$; $\Delta f = f(96) - f(100) = \sqrt{96} - 10$,

$df = f'(100) \cdot \Delta x = \frac{1}{20} \cdot (-4) = \frac{-1}{5}$; assume

$\Delta f \approx df \to$

$$\sqrt{96} - 10 \approx \frac{-1}{5} \longrightarrow \sqrt{96} \approx 9.8$$

10.) (7 pts.) The radius and height of a cylinder are both equal to x so that the volume of the cylinder is given by $V = \pi x^3$. Assume that x is measured with an absolute percentage error of at most 3%. Use a differential to estimate the maximum absolute percentage error in computing the cylinder's volume.

assume $\dfrac{|\Delta x|}{x} \leq 3\%$; find $\dfrac{|\Delta V|}{V}$:

$V = \pi x^3 \longrightarrow V' = 3\pi x^2$, then

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta x|}{V} = \frac{3\cancel{\pi} x^2 \cdot |\Delta x|}{\cancel{\pi} x^3}$$

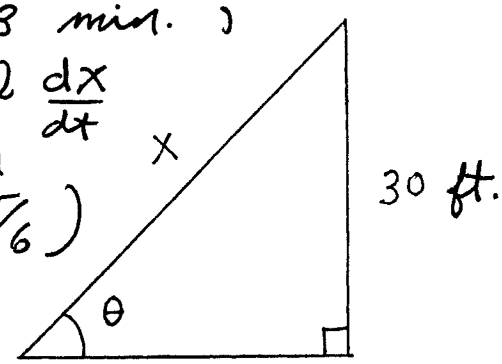
$$= 3 \cdot \frac{|\Delta x|}{x} \leq 3\,(3\%) = 9\%$$

11.) (7 pts.) Consider the given right triangle. If θ increases at the rate of 1/3 radians per minute, at what rate is x changing when $\theta = \pi/6$?

$\sin\theta = \dfrac{30}{x} \xrightarrow{D}$ $\left(\dfrac{d\theta}{dt} = \dfrac{1}{3} \dfrac{\text{rad.}}{\text{min.}} \right.$; Find $\dfrac{dx}{dt}$ when $\left. \theta = \pi/6 \right)$

$\cos\theta \cdot \dfrac{d\theta}{dt} = \dfrac{-30}{x^2} \cdot \dfrac{dx}{dt}$

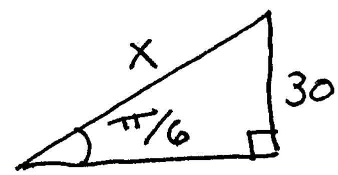
(Let $\theta = \dfrac{\pi}{6}$)



$\rightarrow \cos\dfrac{\pi}{6} \cdot \left(\dfrac{1}{3}\right) = \dfrac{-30}{60^2} \cdot \dfrac{dx}{dt}$

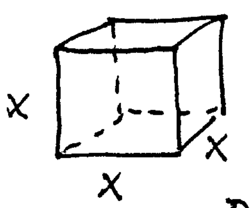
$\rightarrow \dfrac{\sqrt{3}}{2} \cdot \dfrac{1}{3} = \dfrac{-1}{120} \dfrac{dx}{dt}$

$\rightarrow \dfrac{dx}{dt} = \dfrac{-60\sqrt{3}}{3}$ ft./min.



$\sin\dfrac{\pi}{6} = \dfrac{1}{2} = \dfrac{30}{x} \rightarrow x = 60$

12.) (7 pts.) The surface area of a cube is increasing at the rate of 6 ft.^{2}/min. Determine the rate at which the volume of the cube is changing when the edge of the cube is 2 ft.



$S = 6x^2$, $V = x^3$; assume $\dfrac{dS}{dt} = 6 \dfrac{\text{ft.}^2}{\text{min}}$; find $\dfrac{dV}{dt}$ when $x = 2$ ft. :

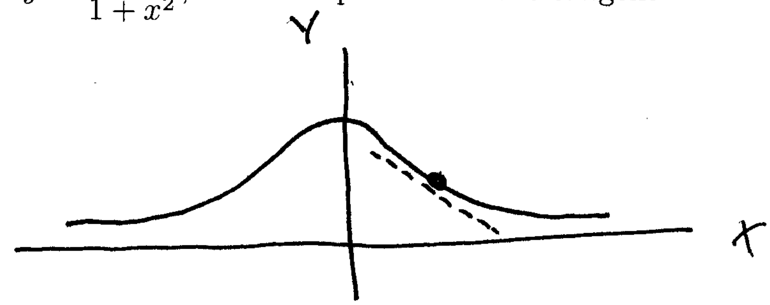
$S = 6x^2 \xrightarrow{D} \dfrac{dS}{dt} = 12x \dfrac{dx}{dt} \rightarrow 6 = 12(2) \dfrac{dx}{dt} \rightarrow$

$\boxed{\dfrac{dx}{dt} = \dfrac{1}{4} \text{ ft./min}}$; $V = x^3 \xrightarrow{D} \dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt}$

$= 3(2)^2\left(\dfrac{1}{4}\right) = 3$ ft.^{3}/min.

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Among all tangent lines to the graph of $y = \dfrac{1}{1+x^2}$, find an equation for the tangent line of minimum slope.



$$Y = \frac{1}{1+X^2} \xrightarrow{D} (\text{SLOPE})$$

$$Y' = \frac{-2X}{(1+X^2)^2};$$

Find MIN SLOPE : $\xrightarrow{D}$

$$Y'' = \frac{(1+X^2)^2(-2) - (-2X)\cdot 2(1+X^2)(2X)}{(1+X^2)^4}$$

$$= \frac{-2(1+X^2)\left[(1+X^2) - 4X^2\right]}{(1+X^2)^4}$$

$$= \frac{-2\left[1-3X^2\right]}{(1+X^2)^3} = 0$$

$+ \quad 0 \quad - \quad 0 \quad +$ $\quad Y''$

$X = -1/\sqrt{3}$, $Y = 3/4$

$X = 1/\sqrt{3}$, $Y = 3/4$ } MIN

$$Y' = \frac{-2/\sqrt{3}}{16/9} = \frac{-9}{8\sqrt{3}}$$

so TANGENT LINE is

$$Y - \frac{3}{4} = \frac{-9}{8\sqrt{3}}\left(X - \frac{1}{\sqrt{3}}\right)$$