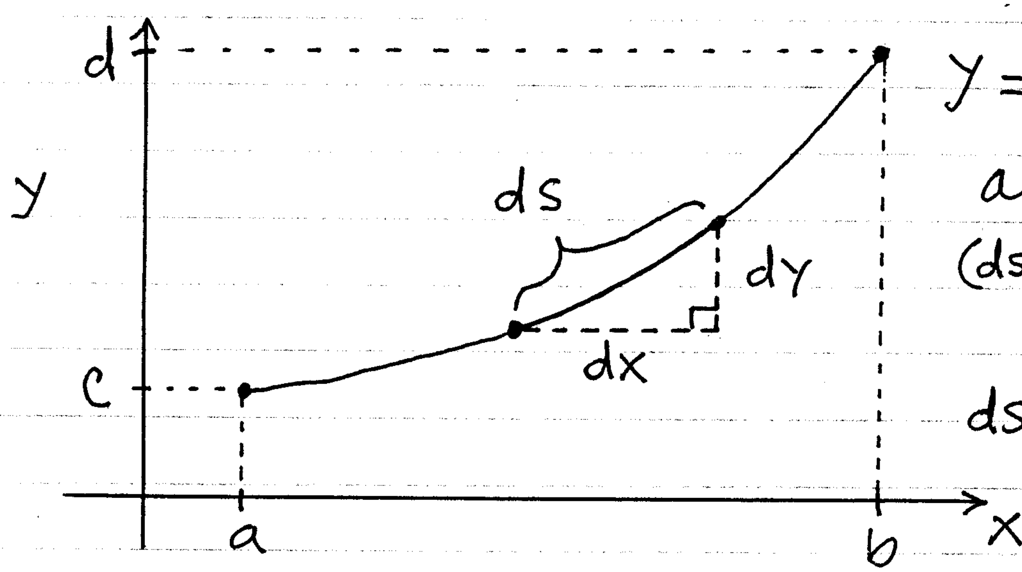
Math 21B (Kouba)
Arc Length

Assume $Y = f(x)$ is a continuous function from $x=a$ to $x=b$. We seek a formula for the length of its graph from $x=a$ to $x=b$.



$Y = f(x)$

assume that $(ds)^2 = (dx)^2 + (dY)^2$ so that $ds = \sqrt{(dx)^2 + (dY)^2}$, then

arc length from $x=a$ to $x=b$ is

$$ARC = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^b \sqrt{\left(1 + \frac{(dY)^2}{(dx)^2}\right)(dx)^2}$$

$$= \int_a^b \sqrt{1 + \left(\frac{dY}{dx}\right)^2}\, dx \,, \quad \text{i.e.,}$$

$$\boxed{ARC = \int_a^b \sqrt{1 + \left(\frac{dY}{dx}\right)^2}\, dx} \;; \quad \text{OR}$$

arc length from $y=c$ to $y=d$ is

$$ARC = \int_c^d ds = \int_c^d \sqrt{(dx)^2+(dy)^2}$$

$$= \int_c^d \sqrt{\left(\frac{(dx)^2}{(dy)^2}+1\right)(dy)^2}$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad , \text{ i.e.,}$$

$$\boxed{ARC = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$