Math 21B (Kouba)
Comparison Tests for Improper Integrals
Absolute Convergence Test

<u>Comparison Test for Convergence</u>
Assume f and g are continuous functions with $0 \le f(x) \le g(x)$. If $\int_a^\infty g(x)\,dx$ converges (finite), then $\int_a^\infty f(x)\,dx$ converges.

<u>Ex</u>: $\displaystyle\int_2^\infty \frac{2x+1}{x^5+x^4+3}\,dx \le \int_2^\infty \frac{2x+x}{x^5+0+0}\,dx$

$\displaystyle = \int_2^\infty \frac{3x}{x^5}\,dx = 3\int_2^\infty \frac{1}{x^4}\,dx = 3\cdot\lim_{A\to\infty}\int_2^A x^{-4}\,dx$

$\displaystyle = 3\cdot\lim_{A\to\infty} \frac{-1}{3x^3}\Big|_2^A = 3\cdot\lim_{A\to\infty}\left\{\frac{-1}{3A^3} - \frac{-1}{24}\right\} = \frac{1}{24} < \infty,$

so $\displaystyle\int_2^\infty \frac{2x+1}{x^5+x^4+3}\,dx$ converges.

<u>Comparison Test for Divergence</u>:
Assume f and g are continuous functions with $0 \le g(x) \le f(x)$. If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

<u>Ex</u>: $\displaystyle\int_1^\infty \frac{x^3+3}{x^4+1}\,dx \ge \int_1^\infty \frac{x^3+0}{x^4+x^4}\,dx$

$\displaystyle = \int_1^\infty \frac{x^3}{2x^4}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{x}\,dx = \frac{1}{2}\lim_{A\to\infty}\int_1^A \frac{1}{x}\,dx$

$\displaystyle = \frac{1}{2}\lim_{A\to\infty} \ln|x|\Big|_1^A = \frac{1}{2}\lim_{A\to\infty}(\ln A - \underbrace{\ln 1}_{0}) = \infty,$

so $\displaystyle\int_1^\infty \frac{x^3+3}{x^4+1}\,dx$ diverges.

Absolute Convergence Test:
Assume f is any continuous function with positive or negative values. If $\int_a^{\infty} |f(x)|\,dx$ converges, then

$$\int_a^{\infty} f(x)\,dx \text{ converges.}$$

Ex: Consider $\int_3^{\infty} \frac{\sin^3 7x}{x^2+9}\,dx$; since

$$\int_3^{\infty} \left|\frac{\sin^3 7x}{x^2+9}\right| dx = \int_3^{\infty} \frac{|\sin^3 7x|}{x^2+9}\,dx$$

$$= \int_3^{\infty} \frac{|\sin 7x|^3}{x^2+9}\,dx \leq \int_3^{\infty} \frac{(1)^3}{x^2+3^2}\,dx$$

$$= \lim_{A\to\infty} \int_3^{A} \frac{1}{x^2+3^2}\,dx = \lim_{A\to\infty} \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_3^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{3}\arctan\left(\frac{A}{3}\right) - \frac{1}{3}\arctan 1\right)$$

$$= \lim_{A\to\infty} \left(\frac{1}{3}\arctan\left(\frac{A}{3}\right) - \frac{1}{3}\cdot\frac{\pi}{4}\right)$$

$$= \frac{1}{3}\cdot\arctan(\infty) - \frac{\pi}{12} = \frac{1}{3}\cdot\left(\frac{\pi}{2}\right) - \frac{\pi}{12} = \frac{\pi}{12} < \infty,$$

it follows that $\int_3^{\infty} \left|\frac{\sin^3 7x}{x^2+9}\right| dx$ converges by the Comparison Test so that $\int_3^{\infty} \frac{\sin^3 7x}{x^2+9}\,dx$ converges by the Absolute Convergence Test.