Math 21B
Kouba
Estimating the Value of a Definite Integral

Suppose that the integral $\int_a^b f(x)\,dx$ is too difficult (or impossible) to compute, or that you are simply required to estimate its exact value. The following three methods offer three different ways to compute an estimate.

1.) MIDPOINT RULE

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval and let the sampling points $c_1, c_2, c_3, ..., c_n$ be the MIDPOINTS of these subintervals.

c.) The Midpoint Estimate for $\int_a^b f(x)\,dx$ is

$$M_n = h\ [f(c_1) + f(c_2) + f(c_3) + ... + f(c_n)] \ .$$

d.) The Absolute Error is $|E_n| \le (b-a)\dfrac{h^2}{24}\left\{ \max_{a \le x \le b} |f''(x)| \right\}$.

2.) TRAPEZOIDAL RULE

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Trapezoidal Estimate for $\int_a^b f(x)\,dx$ is

$$T_n = \frac{h}{2}\ [f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)] \ .$$

d.) The Absolute Error is $|E_n| \le (b-a)\dfrac{h^2}{12}\left\{ \max_{a \le x \le b} |f''(x)| \right\}$.

3.) SIMPSON'S RULE (NOTE: For this method n MUST be an even integer !)

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Simpson Estimate for $\int_a^b f(x)\,dx$ is

$$S_n = \frac{h}{3}\ [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \ .$$

d.) The Absolute Error is $|E_n| \le (b-a)\dfrac{h^4}{180}\left\{ \max_{a \le x \le b} |f^{(4)}(x)| \right\}$.

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1.) Assume that $f''(x) = x^2 e^{3x}$. Find a number M so that $\max\limits_{0 \le x \le 2} |f''(x)| \le M$.

2.) Assume that $f''(x) = \dfrac{7}{2x+3}$. Find a number M so that $\max\limits_{-1 \le x \le 1} |f''(x)| \le M$.

3.) Assume that $f^{(4)}(x) = \dfrac{x-3}{5-x}$. Find a number M so that $\max\limits_{-2 \le x \le 3} |f^{(4)}(x)| \le M$.

4.) Consider the definite integral $\displaystyle\int_{-2}^{2} \sqrt{x^2+4}\, dx$.

a.) Find M_4 , the Midpoint Estimate using $n = 4$.
b.) What should n be in order that the Midpoint Estimate M_n estimate the exact value of this definte integral with absolute error at most 0.0001 ?

5.) Consider the definite integral $\displaystyle\int_{0}^{4} \ln(x^2+1)\, dx$.

a.) Find T_4 , the Trapezoidal Estimate using $n = 4$.
b.) What should n be in order that the Trapezoidal Estimate T_n estimate the exact value of this definte integral with absolute error at most 0.0001 ?

6.) Consider the definite integral $\displaystyle\int_{0}^{2} \frac{x+1}{x+3}\, dx$.

a.) Find S_4 , the Simpson Estimate using $n = 4$.
b.) What should n be in order that the Simpson Estimate S_n estimate the exact value of this definte integral with absolute error at most 0.0001 ?

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Simpson's Rule

Example:

Use S_4, Simpson's Rule with $n=4$, to estimate the exact value of $\displaystyle\int_{-5}^{-4} \frac{x+1}{x+3}\,dx$.

$f(x) = \dfrac{x+1}{x+3}$, $n=4$

$-5 \quad \frac{-19}{4} \quad \frac{-9}{2} \quad \frac{-17}{4} \quad -4$

$$S_4 = \frac{-4-(-5)}{3(4)}\left[f(-5) + 4f\left(\tfrac{-19}{4}\right) + 2f\left(\tfrac{-9}{2}\right) + 4f\left(\tfrac{-17}{4}\right) + f(-4)\right]$$
$$= \frac{1}{12}\left[2 + 4\left(\tfrac{15}{7}\right) + 2\left(\tfrac{7}{3}\right) + 4\left(\tfrac{13}{5}\right) + 3\right] \approx 2.3865 \;;$$

exact value: $\displaystyle\int_{-5}^{-4} \frac{x+1}{x+3}\,dx = 1 + \ln 4 \approx 2.3863$;

absolute error $\displaystyle |E_4| = \left| \int_{-5}^{-4} \frac{x+1}{x+3}\,dx - S_4 \right| = 0.0002$

Question: What should n be in order that S_n, Simpson's Rule with n subdivisions, estimate the exact value of $\displaystyle\int_{-5}^{-4} \frac{x+1}{x+3}\,dx$ with absolute error at most 0.00001 ?

absolute error $\displaystyle |E_n| \le \frac{(b-a)^5}{180\,n^4} \cdot \max_{a \le x \le b} |f^{(4)}(x)| \rightarrow$

$f(x) = \dfrac{x+1}{x+3}$, $f'(x) = 2(x+3)^{-2}$, $f''(x) = -4(x+3)^{-3}$, $f'''(x) = 12(x+3)^{-4}$, $f^{(4)}(x) = \dfrac{-48}{(x+3)^5}$ so

$\displaystyle\max_{-5 \le x \le -4} |f^{(4)}(x)| = \frac{48}{|(-4)+3|^5} = 48$; then

$$|E_n| \le \frac{(-4-(-5))^5}{180\,n^4} \cdot 48 = \frac{12}{45n^4} \le 0.00001 \rightarrow$$

$$n^4 \ge \frac{12}{45(0.00001)} \rightarrow n \ge \left[\frac{12}{45(0.00001)}\right]^{1/4} \approx 12.7$$ so use $n = 14$!!