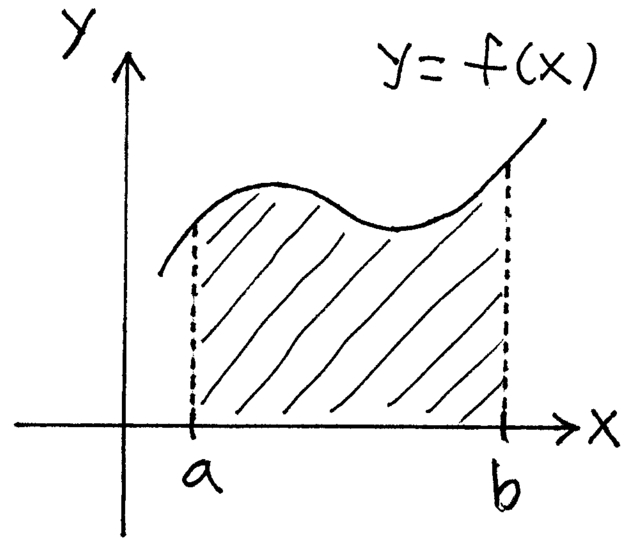
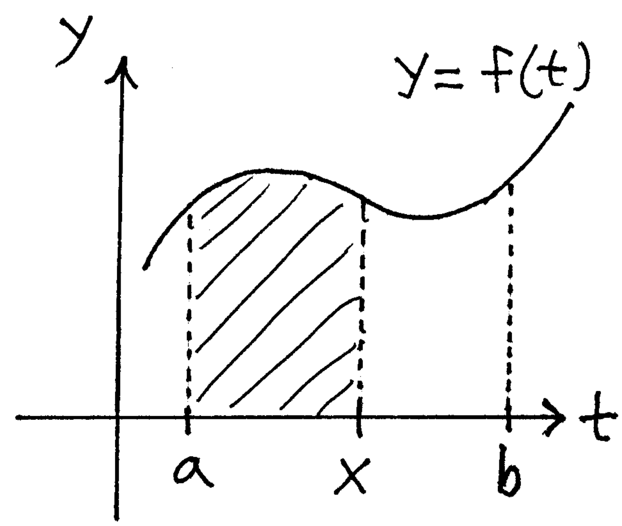
Math 21B
Kouba
FTC1, FTC2 (Making the Connection Between area and antiderivatives)




RECALL: If $f(x) \geq 0$, then
$$\int_a^b f(x)\,dx = \text{Area}$$
of shaded region.




Define a new "partial area" function
$$F(x) = \int_a^x f(t)\,dt$$

NOTE:

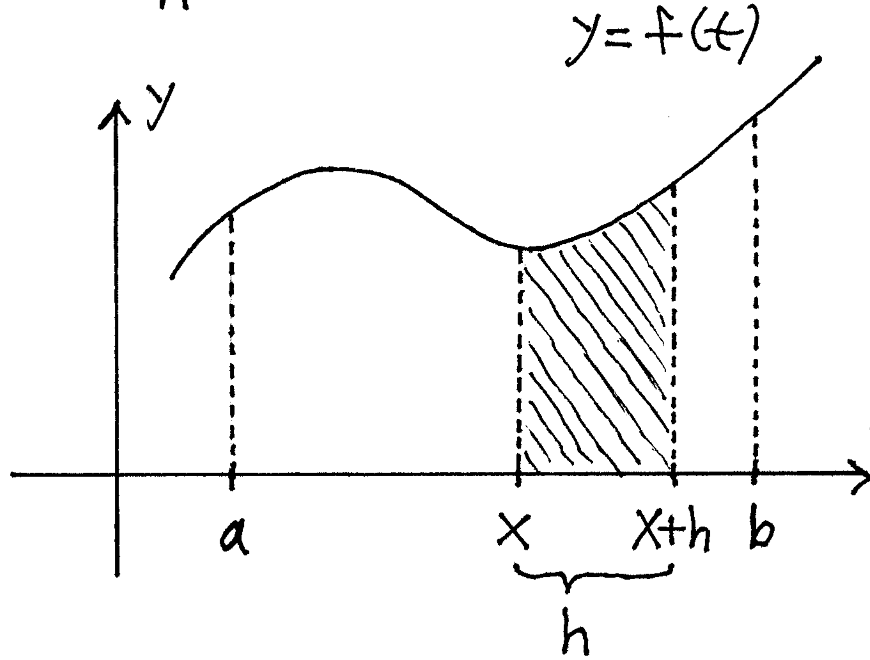
1.) $F(b) = \int_a^b f(t)\,dt$

2.) $F(a) = \int_a^a f(t)\,dt = 0$

Find the derivative of $F(x)$:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

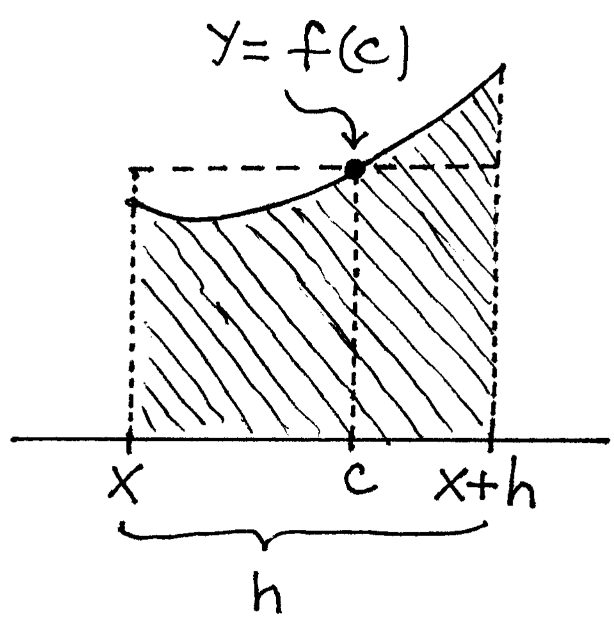
$$= \lim_{h \to 0} \frac{\text{Area Thin Strip } (x \text{ to } x+h)}{h}$$




(By the Mean Value Theorem for Integrals there is some # c, $x \le c \le x+h$, so that

$$\underbrace{f(c) \cdot h}_{\text{Area of Rectangle}} = \text{Area Thin Strip}$$

)




$$= \lim_{h \to 0} \frac{f(c) \cdot \cancel{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} f(c)$$

$= f(x)$, i.e., $F'(x) = f(x)$.

This means that the "partial area" function is an <u>anti-derivative</u> of $f(x)$. We have proven the following theorem.

$\boxed{\text{FTC 1}}$: 1.) $D\int_a^x f(t)\,dt = f(x)$

2.) (Chain Rule) $D\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$

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Let $A(x)$ be any antiderivative of $f(x)$, then we know (from MVT) that $F(x) = A(x) + C$. Thus

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^b f(t)\,dt \\ &= F(b) \\ &= F(b) - 0 \\ &= F(b) - F(a) \\ &= (A(b) + \cancel{C}) - (A(a) + \cancel{C}) = A(b) - A(a).\end{aligned}$$

$\boxed{\text{FTC 2}}$: $\int_a^b f(x)\,dx = A(x)\Big|_a^b = A(b) - A(a)$