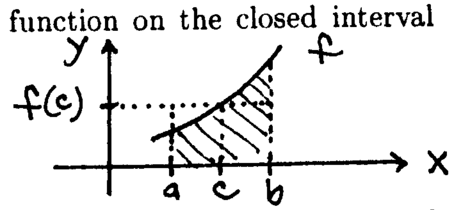
Math 21B
Kouba
MVT and FTC's

Mean Value Theorem for Integrals : If f is a continuous function on the closed interval $[a, b]$, then there is at least one number c , $a \leq c \leq b$, so that

$$f(c)(b-a) = \int_a^b f(x)\, dx \ .$$

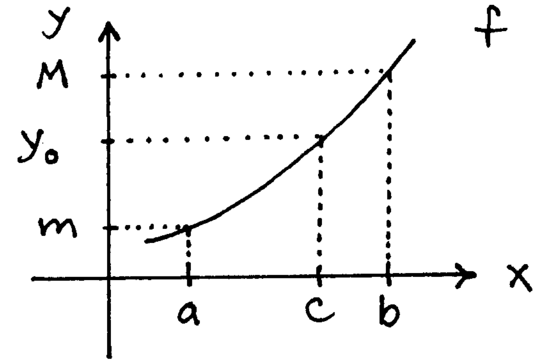


Proof : Since f is a continuous function on the closed interval $[a, b]$, by the Maximum- and Minimum-Value Theorems, f has a maximum vaule M and a minimum value m on $[a, b]$, i.e., $m \leq f(x) \leq M$ on $[a, b]$. By property 6.) (p. 317) of definite integrals,

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a) \ ,$$

so that

$$m \leq \underbrace{\frac{1}{b-a} \int_a^b f(x)\, dx}_{\text{call this number } y_o} \leq M \ ,$$

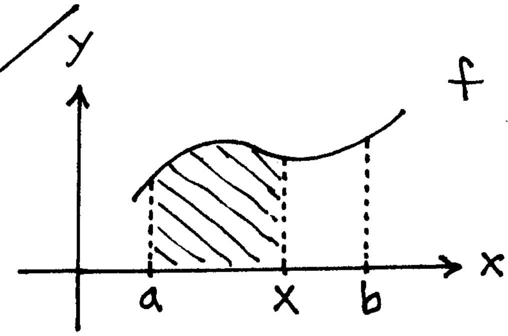


By the Intermediate Value Theorem (p. 99) there is at least one number c , $a \leq c \leq b$, so that

$$f(c) = y_o, \text{ i.e., } f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx \ ,$$

so that

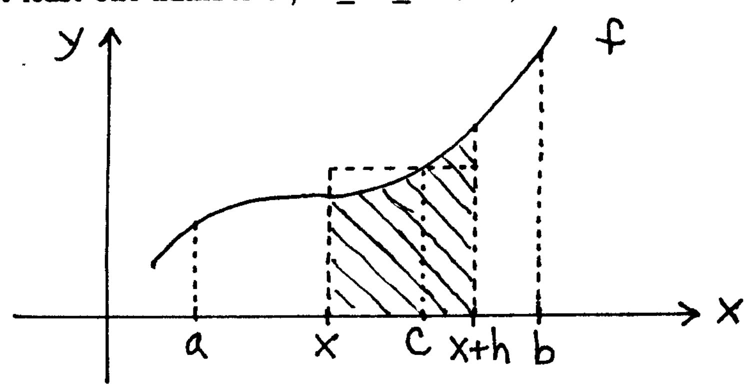
$$f(c)(b-a) = \int_a^b f(x)\, dx \ .$$



First Fundamental Theorem of Calculus (FTC1) : Assume that f is a continuous function on the closed interval $[a, b]$ and that $F(x) = \int_a^x f(t)\, dt$. Then $F'(x) = f(x)$.

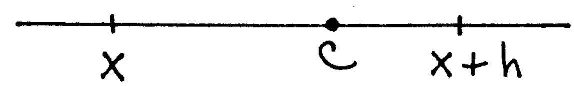
Proof : Consider $F(x) = \int_a^x f(t)\, dt$ as the area under the graph of f above the interval $[a, x]$. Then $F(x+h)$ is the area under the graph of f above the interval $[a, x+h]$ and $F(x+h) - F(x)$ is the area of the " thin strip " from x to $x+h$, i.e., $F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt$. By the Mean Value Theorem for integrals there is at least one number c , $x \leq c \leq x+h$, so that

$$f(c) \cdot h = \int_x^{x+h} f(t)\, dt$$



The derivative of $F(x)$ can now be computed as

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \\
&= \lim_{h\to 0} \frac{f(c)\,h}{h} \\
&= \lim_{h\to 0} f(c) \qquad \text{(Recall that } x \le c \le x+h.\text{)} \\
&= f(x)\ .
\end{aligned}
$$



Second Fundamental Theorem of Calculus (FTC2) : Let f be a continuous function on the closed interval $[a,b]$. Assume that $F(x)$ is an antiderivative of $f(x)$, i.e., assume that $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)\ .$$

Proof : Let $A(x) = \displaystyle\int_a^x f(t)\,dt$. Then $A(a) = 0$, $A(b) = \displaystyle\int_a^b f(t)\,dt$, and $A'(x) = f(x)$ by FTC1. But $F'(x) = f(x)$. By Corollary 2 (p. 233) to the Mean Value Theorem $F(x) = A(x) + C$ for any constant C, or

$$A(x) = F(x) - C\ .$$

Then

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b f(t)\,dt \\
&= A(b) \\
&= A(b) - A(a) \\
&= (F(b)-C) - (F(a)-C) \\
&= F(b) - F(a) \\
&= F(x)\Big|_a^b\ .
\end{aligned}
$$