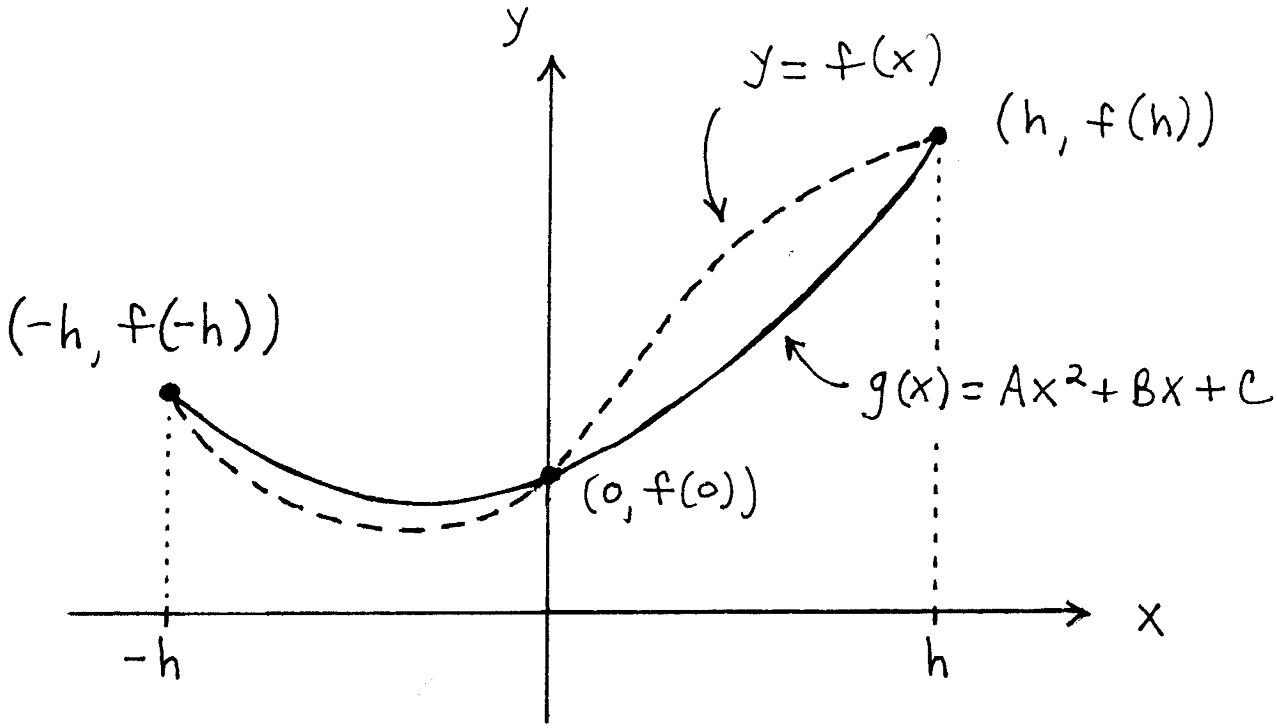
Math 21B
Kouba
Simpson's Rule– The Plausibilty of Its Formula

Assume that function $y = f(x)$ is defined on the closed interval $[-h, h]$. Consider the following three points on the graph of $y = f(x)$:

$$P_1 = (-h, f(-h)),\ P_2 = (0, f(0)),\ \text{and}\ \ P_3 = (h, f(h))$$



A general formula for a parabola is $g(x) = Ax^2 + Bx + C$. Determine the values of the constants A, B, and C for that parabola which passes through the points P_1, P_2, and P_3. It follows immediately that

$$g(0) = f(0)\ ,\ g(-h) = f(-h)\ ,\ \text{and}\ g(h) = f(h)$$

i.e.,

(i) (at $x = 0$) : $C = f(0)$,
(ii) (at $x = h$) : $Ah^2 + Bh + f(0) = f(h)$, and
(iii) (at $x = -h$) : $Ah^2 - Bh + f(0) = f(-h)$.

Adding equations (ii) and (iii) leads to :

$$2Ah^2 + 2f(0) = f(h) + f(-h) \quad \longrightarrow$$
$$2Ah^2 = f(-h) - 2f(0) + f(h) \quad \longrightarrow$$
$$A = \frac{1}{2h^2}[f(-h) - 2f(0) + f(h)] \ .$$

Equation (iii) leads to :

$$Bh = Ah^2 + f(0) - f(-h) \quad \longrightarrow$$

$$B = \frac{1}{h}[Ah^2 + f(0) - f(-h)] \quad \longrightarrow$$

$$= \frac{1}{h}[\frac{1}{2h^2}[f(-h) - 2f(0) + f(h)]h^2 + f(0) - f(-h)] \quad \longrightarrow$$

$$= \frac{1}{h}[\frac{1}{2}f(-h) - f(0) + \frac{1}{2}f(h) + f(0) - f(-h)] \quad \longrightarrow$$

$$= \frac{1}{h}[\frac{1}{2}f(h) - \frac{1}{2}f(-h)] \quad \longrightarrow$$

$$= \frac{1}{2h}[f(h) - f(-h)] \; .$$

The unknown constants A, B, and C are now determined. Consequently, we would expect the following to be true :

$$\int_{-h}^{h} f(x)\,dx \approx \int_{-h}^{h} g(x)\,dx$$

$$= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= (A(1/3)x^3 + B(1/2)x^2 + Cx)\Big|_{-h}^{h}$$

$$= ((1/3)Ah^3 + (1/2)Bh^2 + Ch) - ((-1/3)Ah^3 + (1/2)Bh^2 - Ch)$$

$$= (2/3)Ah^3 + 2Ch$$

$$= (2/3)\frac{1}{2h^2}[f(-h) - 2f(0) + f(h)]h^3 + 2f(0)h$$

$$= \frac{h}{3}[f(-h) + \frac{4h}{3}f(0) + \frac{h}{3}f(h)]$$

$$= \frac{h}{3}[f(-h) + 4f(0) + f(h)] \; .$$