Math 21B (Winter 2017)
Kouba
Exam 2

Please PRINT your name here : KEY

Your Exam ID Number ________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER PERSON TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

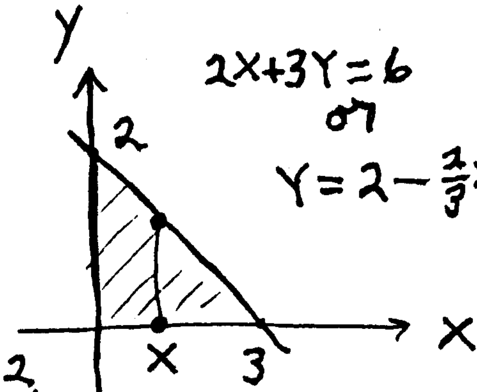
5. Make sure that you have 7 pages, including the cover page.

6. You will be graded on proper use of integral and derivative notation.

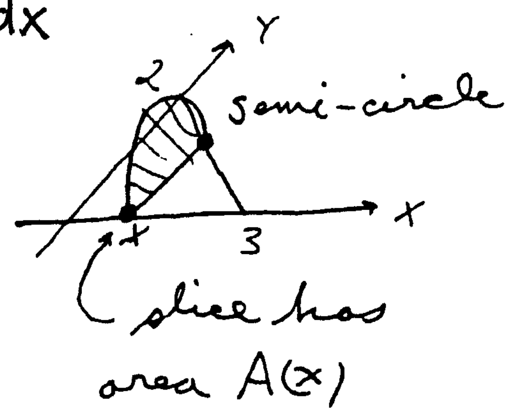
7. You have until 8:50 a.m. to finish the exam. PLEASE STOP WRITING IMMEDIATELY WHEN TIME IS CALLED AND REMAIN SITTING. THANK YOU.

1.) (12 pts.) The base of a solid lies in the region bounded by the graphs of $y = 0$, $x = 0$, and $2x + 3y = 6$. Cross-sections of the solid taken perpendicular to the x-axis at x are *semi-circles*. SET UP BUT DO NOT EVALUATE an integral which represents the *volume* of this solid.

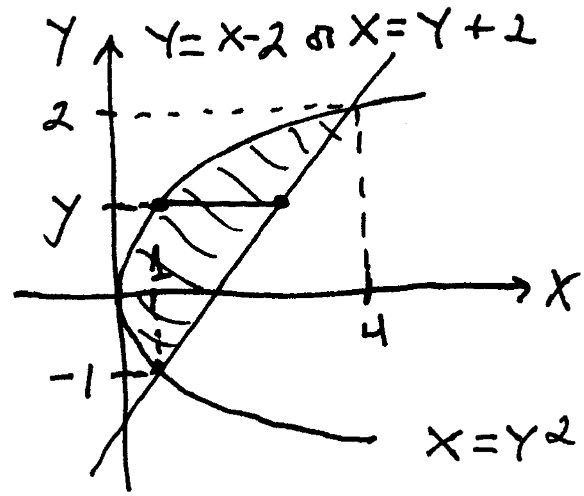
$$A(x) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1}{2}\left(2 - \frac{2}{3}x\right)\right)^2$$
$$= \frac{1}{2}\pi\left(1 - \frac{1}{3}x\right)^2 \quad \text{so}$$



$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 \frac{1}{2}\pi\left(\frac{1}{2}\left(2 - \frac{2}{3}x\right)\right)^2 dx$$
$$= \int_0^3 \frac{1}{2}\pi\left(1 - \frac{1}{3}x\right)^2 dx$$



2.) (12 pts.) A flat plate of *constant* density k lies in the region bounded by the graphs of $x = y^2$ and $y = x - 2$. SET UP BUT DO NOT EVALUATE integrals which represent $(\bar{x}, \bar{y})$, the *centroid* of this plate.



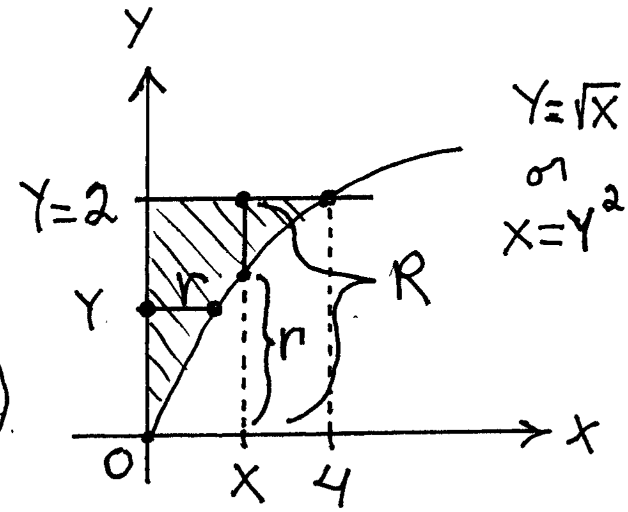
$$\left.\begin{aligned} X &= Y^2 \\ Y &= X - 2 \end{aligned}\right\} \; Y = Y^2 - 2 \rightarrow$$
$$0 = Y^2 - Y - 2 \rightarrow$$
$$0 = (Y-2)(Y+1) \rightarrow$$
$$Y = 2,\ Y = -1$$

$$\bar{X} = \frac{\frac{1}{2}\int_{-1}^{2}\left[(Y+2)^2 - (Y^2)^2\right]dy}{\int_{-1}^{2}\left[(Y+2) - Y^2\right]dy},$$

$$\bar{Y} = \frac{\int_{-1}^{2} Y\left[(Y+2) - Y^2\right]dy}{\int_{-1}^{2}\left[(Y+2) - Y^2\right]dy}$$

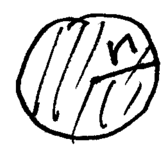
3.) (7 pts. each) Consider the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and $x = 0$. SET UP BUT DO NOT EVALUATE integrals which represent the *volume* of the solid formed by revolving this region about

a.) the x-axis using the DISC METHOD.



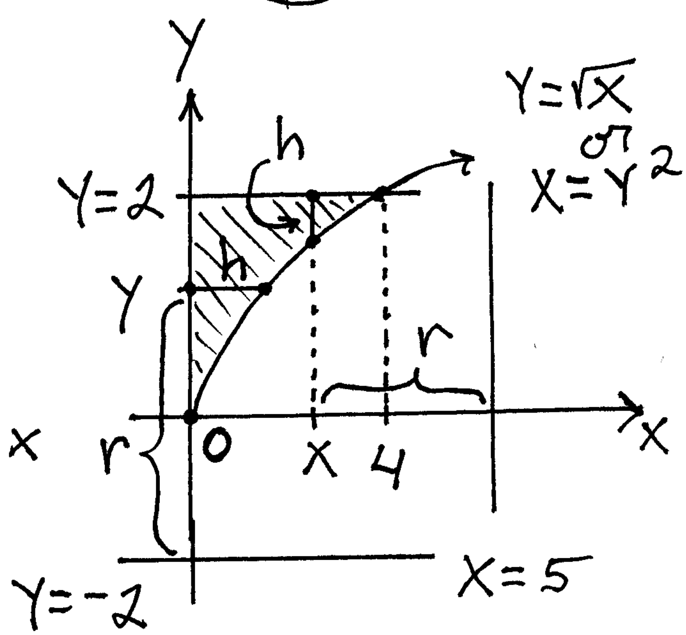
$$Vol = \pi \int_0^4 (2)^2 dx - \pi \int_0^4 (\sqrt{x})^2 dx$$

b.) the y-axis using the DISC METHOD.



$$Vol = \pi \int_0^2 (Y^2)^2 dy$$

c.) the line $x = 5$ using the SHELL METHOD.



$$Vol = 2\pi \int_0^4 (5-x)(2-\sqrt{x}) dx$$

d.) the line $y = -2$ using the SHELL METHOD.

$$Vol = 2\pi \int_0^2 (Y+2)(Y^2) dy$$

4.) (13 pts.) Set up and EVALUATE an integral which represents the *arc length* of the curve given by the equation $y = (1/6)x^3 + \frac{1}{2x}$ on the interval $1/2 \le x \le 1$.

$$y' = \frac{1}{6}\cdot 3x^2 - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2} \quad \text{so}$$

$$\text{Arc} = \int_{1/2}^{1} \sqrt{1+(y')^2}\, dy = \int_{1/2}^{1} \sqrt{1+\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$$

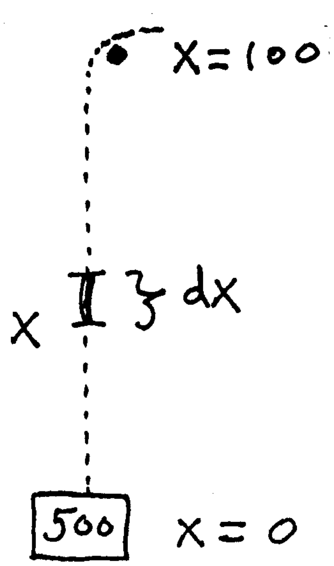
$$= \int_{1/2}^{1} \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\, dx$$

$$= \int_{1/2}^{1} \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\, dx = \int_{1/2}^{1} \sqrt{\frac{x^8+2x^4+1}{4x^4}}\, dx$$

$$= \int_{1/2}^{1} \frac{\sqrt{(x^4+1)^2}}{2x^2}\, dx = \int_{1/2}^{1} \frac{x^4+1}{2x^2}\, dx = \int_{1/2}^{1} \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right) dx$$

$$= \left(\frac{1}{6}x^3 - \frac{1}{2x}\right)\Big|_{1/2}^{1} = \left(\frac{1}{6} - \frac{1}{2}\right) - \left(\frac{1}{48} - 1\right) = \frac{8}{48} - \frac{24}{48} - \frac{1}{48} + \frac{48}{48} = \frac{31}{48}$$

5.) (10 pts.) A chain weighs 2 pounds per foot and is used to raise a large bucket on the ground full of sand weighing 500 pounds to a point 100 feet above the ground. As the bucket is raised it loses 3 pounds of sand per foot. SET UP BUT DO NOT EVALUATE an integral which represents the *work* required to complete the task. (Include the weight of the chain in your solution.)



Estimate for work from x to $x+dx$ is

(weight) (distance)

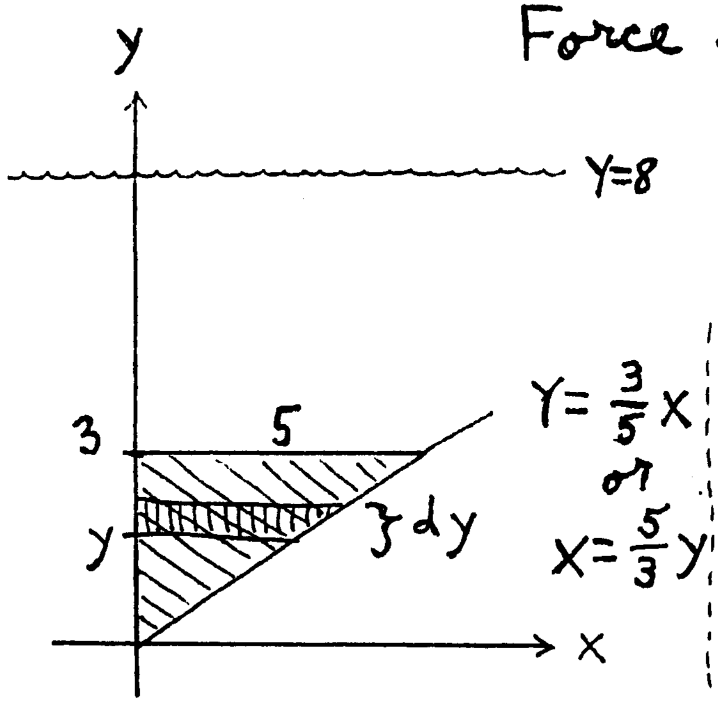
$$\approx \Big(\underbrace{500 - 3x}_{\text{sand}} + \underbrace{2(100-x)}_{\text{chain}}\Big)\cdot dx,$$

so total

$$\text{Work} = \int_0^{100} \big(500 - 3x + 2(100-x)\big)\, dx$$

ft.-lbs.

6.) (10 pts.) A flat plate is in the shape of the given right triangle. It rests (as given in the diagram) at the bottom of a pool filled with water to a depth of 8 feet. SET UP BUT DO NOT EVALUATE an integral which represents the *force* of water pressure on one side of this plate. (Water weighs 62.4 pounds per cubic foot.)



Force on strip

$= (\text{area})(\text{depth})(\text{density})$

$\approx \left(\frac{5}{3}y \cdot dy\right)(8-y)\ (62.4)$,

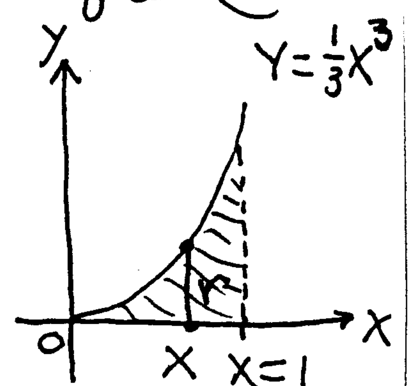
so total

$$\text{Force} = 62.4 \int_0^3 \frac{5}{3}y(8-y)\, dy$$

lbs.

7.) (9 pts.) The graph of $y = (1/3)x^3$ for $0 \le x \le 1$ is rotated about the x-axis. Set up and EVALUATE an integral which represents the *surface area* of this solid of revolution.

$Y = \frac{1}{3}x^3 \xrightarrow{D} Y' = \frac{1}{3} \cdot 3x^2 = x^2$ so surface



$$\text{AREA} = 2\pi \int_0^1 \frac{1}{3}x^3 \cdot \sqrt{1+(Y')^2}\, dx$$

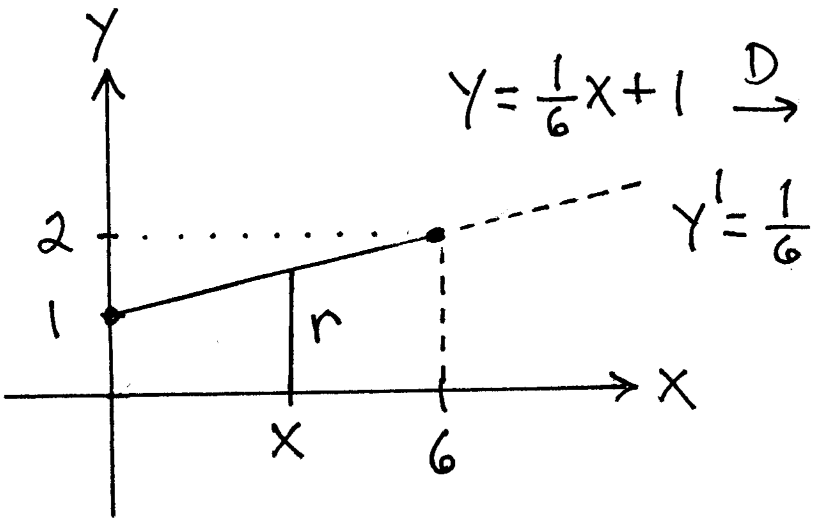
$$= 2\pi \int_0^1 \frac{1}{3}x^3 \sqrt{1+(x^2)^2}\, dx$$

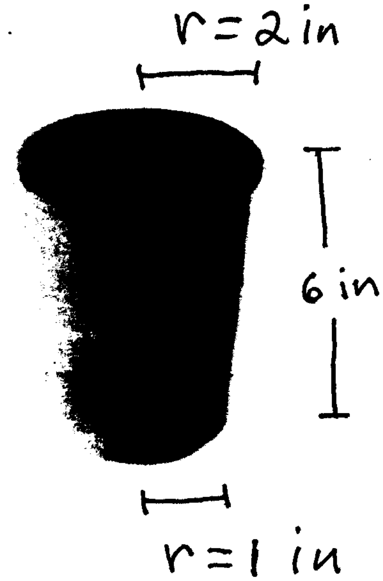
$$= 2\pi \int_0^1 \frac{1}{3}x^3 \sqrt{1+x^4}\, dx$$

$$= 2\pi \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{2}{3} (1+x^4)^{3/2} \Big|_0^1$$

$$= \frac{\pi}{9}(2)^{3/2} - \frac{\pi}{9}$$

8.) (6 pts.) A paper coffee cup is 6 inches tall. Its circular top has circumference 4π inches and its circular bottom has circumference 2π inches. Use calculus to find the surface area of the curved part of the cup.





$$\text{Surface Area} = 2\pi \int_0^6 (\text{radius}) \sqrt{1 + (Y')^2}\, dx$$

$$= 2\pi \int_0^6 \left(\tfrac{1}{6}X + 1\right) \sqrt{1 + \left(\tfrac{1}{6}\right)^2}\, dx$$

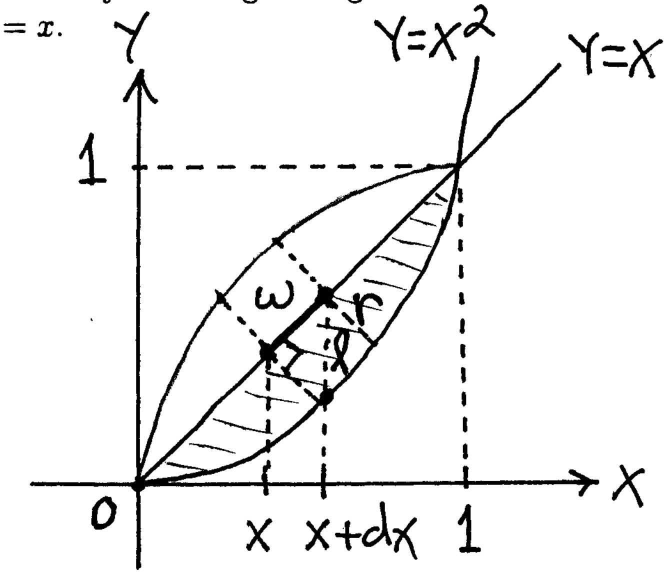
$$= 2\pi \sqrt{\frac{37}{36}} \left[\frac{1}{6} \cdot \frac{1}{2} X^2 + X \right] \Big|_0^6$$

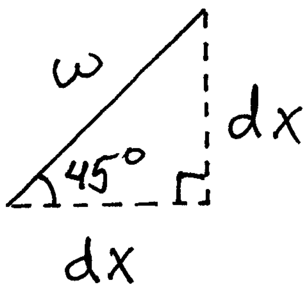
$$= \frac{\pi}{3} \sqrt{37} \left((3+6) - (0+0) \right)$$

$$= 3\pi \sqrt{37} \ \text{in}^2$$

The following EXTRA CREDIT problem is OPTIONAL. It is worth 8 points.

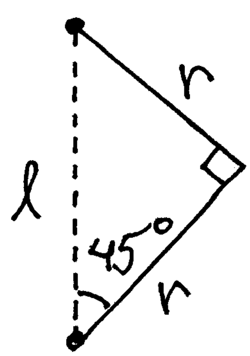
1.) Find the volume of the solid (SET UP ONLY.) formed by revolving the region bounded by the graphs of $y = x$ and $y = x^2$ about the line $y = x$.





$\rightarrow (dx)^2 + (dx)^2 = \omega^2$

$\rightarrow \quad \omega = \sqrt{2}\, dx \quad ;$



$l = x - x^2$ and

$r^2 + r^2 = l^2 \rightarrow$

$2r^2 = l^2 \rightarrow r^2 = \frac{1}{2} l^2 \rightarrow$

$r^2 = \frac{1}{2}(x - x^2)^2 \; ;$

slice taken $\perp$ ~~to~~ $Y = X$ has volume

$\pi r^2 \cdot \omega = \pi \cdot \frac{1}{2}(x - x^2)^2 \cdot \sqrt{2}\, dx$, so

TOTAL VOLUME $= \frac{\pi}{\sqrt{2}} \int_0^1 (x - x^2)^2\, dx$

$$= \frac{\pi}{\sqrt{2}} \int_0^1 (x^2 - 2x^3 + x^4)\, dx$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right] \Big|_0^1$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right]$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{10}{30} - \frac{15}{30} + \frac{6}{30} \right]$$

$$= \frac{\pi}{\sqrt{2}} \cdot \frac{1}{30} \;;$$

$\bar{X} = \frac{1}{2}$, $\bar{Y} = \frac{2}{5}$

pt. $\left(\frac{9}{20}, \frac{9}{20} \right)$