Math 21B (Winter 2017)
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ___________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM SOMEONE ELSE'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE SOMEONE ELSE TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO PARTIAL CREDIT. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 6 pages, including the cover page.
6. You will be graded on proper use of integral and derivative notation.
7. Do not use shortcuts when integrating using the method of integration by parts.
8. Include units on answers where units are appropriate.

9. You have until 8:50 a.m. to finish the exam. PLEASE STOP IMMEDIATELY WHEN TIME IS CALLED. FAILURE TO DO SO MAY LEAD TO POINTS BEING DEDUCTED FROM YOUR EXAM SCORE.

10. You may use the following trig identities :

i.) $\sin(A+B) = \sin A \ \cos B + \cos A \ \sin B$
ii.) $\sin(A-B) = \sin A \ \cos B - \cos A \ \sin B$
iii.) $\cos(A+B) = \cos A \ \cos B - \sin A \ \sin B$
iv.) $\cos(A-B) = \cos A \ \cos B + \sin A \ \sin B$

a.) $\sin^2\theta + \cos^2\theta = 1$
b.) $1 + \tan^2\theta = \sec^2\theta$
c.) $\sin 2\theta = 2\sin\theta\cos\theta$
d.) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

1.) (10 pts. each) Use any method on the following indefinite integrals.

a.) $\int \frac{e^{3x}}{e^{3x}+2}\,dx$ $\left(\text{Let } u = e^{3x} + 2 \xrightarrow{D} du = 3e^{3x}\,dx \rightarrow \frac{1}{3}du = e^{3x}\,dx \right)$

$= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|e^{3x}+2| + C$

b.) $\int x \ln x \, dx$ $\left(\text{Let } u = \ln x,\ dv = x\,dx \rightarrow du = \frac{1}{x}dx,\ v = \frac{1}{2}x^2 \right)$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + C$

c.) $\int \frac{x+1}{x^3 - x^2}\,dx = \int \frac{x+1}{x^2(x-1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \right] dx$

$(Ax(x-1) + B(x-1) + Cx^2 = x + 1$

<u>Let $x=0$</u>: $0 - B + 0 = 1 \rightarrow \boxed{B = -1}$

<u>Let $x=1$</u>: $0 + 0 + C = 2 \rightarrow \boxed{C = 2}$

<u>Let $x=2$</u>: $2A - 1 + 8 = 3 \rightarrow 2A = -4 \rightarrow \boxed{A = -2}$)

$= \int \left[\frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1} \right] dx$

$= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

d.) $\int \frac{\sin^3 x}{\tan^2 x}\,dx = \int \frac{\sin^3 x}{\frac{\sin^2 x}{\cos^2 x}}\,dx = \int \sin x \cdot \cos^2 x\,dx$

(Let $u = \cos x \xrightarrow{D} du = -\sin x\,dx \to$
$-du = \sin x\,dx$)

$= -\int u^2\,du = -\frac{1}{3}u^3 + C = \frac{-1}{3}(\cos x)^3 + C$

e.) $\int \frac{x}{x^2-4x+8}\,dx = \int \frac{x-2+2}{(x-2)^2+4}\,dx$

$= \int \frac{(x-2)}{(x-2)^2+4}\,dx + \int \frac{2}{(x-2)^2+2^2}\,dx$

$= \frac{1}{2}\ln\left|(x-2)^2+4\right| + 2\cdot\frac{1}{2}\arctan\left(\frac{x-2}{2}\right) + C$

f.) $\int \sin 3x\ \cos 2x\,dx$

(Use 10.) i.) + ii.)) $\to$
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$
$\to \sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$)

$= \frac{1}{2}\int [\sin(3x+2x) + \sin(3x-2x)]\,dx$

$= \frac{1}{2}\int [\sin 5x + \sin x]\,dx$

$= \frac{1}{2}\left[\frac{-1}{5}\cos 5x - \cos x\right] + C$

$\boxed{\text{OR}}$

$\int \sin 3x \cos 2x \, dx$ (Let $u = \sin 3x$, $dv = \cos 2x \, dx$
$\rightarrow du = 3\cos 3x$, $v = \frac{1}{2}\sin 2x$)

$= \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\int \cos 3x \sin 2x \, dx$
(Let $u = \cos 3x$, $dv = \sin 2x$
$\rightarrow du = -3\sin 3x$, $v = \frac{-1}{2}\cos 2x$)

$= \frac{1}{2}\sin 3x \sin 2x$
$- \frac{3}{2}\left[\frac{-1}{2}\cos 3x \cos 2x - \frac{3}{2}\int \sin 3x \cos 2x \, dx\right]$

$= \frac{1}{2}\sin 3x \sin 2x$
$+ \frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}\int \sin 3x \cos 2x \, dx$
(Now TWIST) $\longrightarrow$

$-\frac{5}{4}\int \sin 3x \cos 2x \, dx$
$= \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x + C \rightarrow$

$\int \sin 3x \cos 2x \, dx$
$= \frac{-2}{5}\sin 3x \sin 2x - \frac{3}{5}\cos 3x \cos 2x + C$

g.) $\int x^2 \sin x \, dx$ $\quad$ (Let $u = x^2$, $dv = \sin x \, dx \rightarrow$ $du = 2x \, dx$, $v = -\cos x$)

$= -x^2 \cos x - {-2} \int x \cos x \, dx$

(Let $u = x$, $dv = \cos x \, dx \rightarrow$ $du = dx$, $v = \sin x$)

$= -x^2 \cos x + 2 \left[x \sin x - \int \sin x \, dx \right]$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$

h.) $\int \frac{\sqrt{1-x^2}}{x} \, dx$ $\quad$ (Let $x = \sin\theta \xrightarrow{D} dx = \cos\theta \, d\theta$)

$= \int \frac{\sqrt{1-\sin^2\theta} \cdot \cos\theta}{\sin\theta} \, d\theta = \int \frac{\sqrt{\cos^2\theta} \cdot \cos\theta}{\sin\theta} \, d\theta$

$= \int \frac{\cos^2\theta}{\sin\theta} \, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta} \, d\theta$

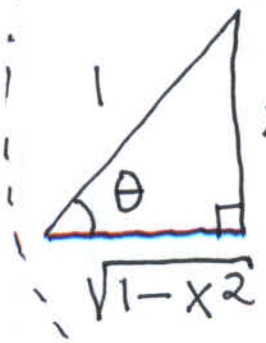
$= \int (\csc\theta - \sin\theta) \, d\theta$

$= \ln|\csc\theta - \cot\theta| + \cos\theta + C$

$= \ln\left| \frac{1}{x} - \frac{\sqrt{1-x^2}}{x} \right|$

$+ \sqrt{1-x^2} + C$

$\sin\theta = \frac{x}{1}$



2.) (10 pts.) Assume that the third derivative of $f(x)$ is $f'''(x) = x + \ln(x^2+1)$. What should n be so that S_n, the Simpson Rule Estimate using n parts, estimates the exact value of the integral $\int_0^2 f(x)\,dx$ with absolute error at most 0.0001 ? Assume that the Absolute Error formula is $|E_n| \le (b-a)\frac{h^4}{180}\left\{\max_{a\le x\le b}|f^{(4)}(x)|\right\}$.

$$\xrightarrow{D} f^{(4)}(x) = 1 + \frac{2x}{x^2+1} = \frac{x^2+1}{x^2+1} + \frac{2x}{x^2+1} = \frac{(x+1)^2}{x^2+1},$$

$$h = \frac{2-0}{n} = \frac{2}{n}, \text{ then}$$

$$|E_n| \le (2-0)\frac{\left(\frac{2}{n}\right)^4}{180}\left\{\max_{0\le x\le 2}|f^{(4)}(x)|\right\}$$

(BIG/SMALL GAME :

$$\max_{0\le x\le 2}|f^4(x)| = \max_{0\le x\le 2}\frac{(x+1)^2}{x^2+1}$$

$$\le \frac{((2)+1)^2}{(0)^2+1} = 9)$$

$$= \frac{32}{180}\cdot\frac{1}{n^4}\{9\} \le 0.0001 \rightarrow$$

$$n^4 \ge \frac{(32)(9)}{(180)(0.0001)} \rightarrow$$

$$n \ge \left[\frac{(32)(9)}{(180)(0.0001)}\right]^{1/4} = [16,000]^{1/4}$$

$$\approx 11.24$$

so choose $\boxed{n = 12}$. (There are many correct answers.)

3.) (10 pts.) A human mummy, discovered recently in the Andes Mountains of Argentina and called La Doncella, is believed to be about 510 years old. How much of the original Carbon 14 (with a half-life of 5730 years) remains in the mummy ? Assume exponential decay of the Carbon 14.

$A = Ce^{kt}$, C: initial amount,

$t = 5730$, $A = \frac{1}{2}C \rightarrow \frac{1}{2}\cancel{C} = \cancel{C}e^{5730k} \rightarrow$

$\ln(1/2) = \ln e^{5730k} = 5730k \rightarrow$

$k = \frac{\ln(1/2)}{5730} \rightarrow \underline{A = Ce^{\frac{\ln(1/2)}{5730}t}}$;

let $t = 510$ yrs. $\rightarrow$

$A = Ce^{\frac{\ln(1/2)}{5730}(510)} \approx C(0.9402)$,

so about 94.02 % of C remains

The following EXTRA CREDIT problem is OPTIONAL. It is worth 8 points.

1.) Integrate : $\displaystyle\int \frac{1 + \cot x}{1 + e^{-x}\csc x}\, dx$

$= \displaystyle\int \frac{1+\cot x}{1+e^{-x}\csc x} \cdot \frac{e^x \sin x}{e^x \sin x}\, dx$

$= \displaystyle\int \frac{e^x \sin x + e^x \cos x}{e^x \sin x + 1}\, dx$

$= \ln\left|e^x \sin x + 1\right| + C$