Exercises 4.8

Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. **a.** $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$
2. **a.** $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$
3. **a.** $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$
4. **a.** $2x^{-3}$ **b.** $\frac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$
5. **a.** $\frac{1}{x^2}$ **b.** $\frac{5}{x^2}$ **c.** $2 - \frac{5}{x^2}$
6. **a.** $-\frac{2}{x^3}$ **b.** $\frac{1}{2x^3}$ **c.** $x^3 - \frac{1}{x^3}$
7. **a.** $\frac{3}{2}\sqrt{x}$ **b.** $\frac{1}{2\sqrt{x}}$ **c.** $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. **a.** $\frac{4}{3}\sqrt[3]{x}$ **b.** $\frac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. **a.** $\frac{2}{3}x^{-1/3}$ **b.** $\frac{1}{3}x^{-2/3}$ **c.** $-\frac{1}{3}x^{-4/3}$
10. **a.** $\frac{1}{2}x^{-1/2}$ **b.** $-\frac{1}{2}x^{-3/2}$ **c.** $-\frac{3}{2}x^{-5/2}$
11. **a.** $\frac{1}{x}$ **b.** $\frac{7}{x}$ **c.** $1 - \frac{5}{x}$
12. **a.** $\frac{1}{3x}$ **b.** $\frac{2}{5x}$ **c.** $1 + \frac{4}{3x} - \frac{1}{x^2}$
13. **a.** $-\pi \sin \pi x$ **b.** $3 \sin x$ **c.** $\sin \pi x - 3 \sin 3x$
14. **a.** $\pi \cos \pi x$ **b.** $\frac{\pi}{2} \cos \frac{\pi x}{2}$ **c.** $\cos \frac{\pi x}{2} + \pi \cos x$
15. **a.** $\sec^2 x$ **b.** $\frac{2}{3} \sec^2 \frac{x}{3}$ **c.** $-\sec^2 \frac{3x}{2}$
16. **a.** $\csc^2 x$ **b.** $-\frac{3}{2} \csc^2 \frac{3x}{2}$ **c.** $1 - 8 \csc^2 2x$
17. **a.** $\csc x \cot x$ **b.** $-\csc 5x \cot 5x$ **c.** $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
18. **a.** $\sec x \tan x$ **b.** $4 \sec 3x \tan 3x$ **c.** $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$
19. **a.** e^{3x} **b.** e^{-x} **c.** $e^{x/2}$
20. **a.** e^{-2x} **b.** $e^{4x/3}$ **c.** $e^{-x/5}$
21. **a.** 3^x **b.** 2^{-x} **c.** $\left(\frac{5}{3}\right)^x$
22. **a.** $x^{\sqrt{3}}$ **b.** x^{π} **c.** $x^{\sqrt{2}-1}$
23. **a.** $\frac{2}{\sqrt{1 - x^2}}$ **b.** $\frac{1}{2(x^2 + 1)}$ **c.** $\frac{1}{1 + 4x^2}$
24. **a.** $x - \left(\frac{1}{2}\right)^x$ **b.** $x^2 + 2^x$ **c.** $\pi^x - x^{-1}$

Finding Indefinite Integrals

In Exercises 25–70, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

25. $\int (x + 1)\, dx$
26. $\int (5 - 6x)\, dx$
27. $\int \left(3t^2 + \frac{t}{2}\right) dt$
28. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
29. $\int (2x^3 - 5x + 7)\, dx$
30. $\int (1 - x^2 - 3x^5)\, dx$
31. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
32. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
33. $\int x^{-1/3}\, dx$
34. $\int x^{-5/4}\, dx$
35. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
36. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$
38. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$
39. $\int 2x(1 - x^{-3})\, dx$
40. $\int x^{-3}(x + 1)\, dx$
41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$
42. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$
43. $\int (-2 \cos t)\, dt$
44. $\int (-5 \sin t)\, dt$
45. $\int 7 \sin \frac{\theta}{3}\, d\theta$
46. $\int 3 \cos 5\theta\, d\theta$
47. $\int (-3 \csc^2 x)\, dx$
48. $\int \left(-\frac{\sec^2 x}{3}\right) dx$
49. $\int \frac{\csc \theta \cot \theta}{2}\, d\theta$
50. $\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$
51. $\int (e^{3x} + 5e^{-x})\, dx$
52. $\int (2e^x - 3e^{-2x})\, dx$
53. $\int (e^{-x} + 4^x)\, dx$
54. $\int (1.3)^x\, dx$
55. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$
56. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$
57. $\int (\sin 2x - \csc^2 x)\, dx$
58. $\int (2 \cos 2x - 3 \sin 3x)\, dx$
59. $\int \frac{1 + \cos 4t}{2}\, dt$
60. $\int \frac{1 - \cos 6t}{2}\, dt$
61. $\int \left(\frac{1}{x} - \frac{5}{x^2 + 1}\right) dx$
62. $\int \left(\frac{2}{\sqrt{1 - y^2}} - \frac{1}{y^{1/4}}\right) dy$
63. $\int 3x^{\sqrt{3}}\, dx$
64. $\int x^{\sqrt{2}-1}\, dx$

65. $\int (1 + \tan^2 \theta)\, d\theta$

66. $\int (2 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

67. $\int \cot^2 x\, dx$

68. $\int (1 - \cot^2 x)\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

69. $\int \cos \theta (\tan \theta + \sec \theta)\, d\theta$

70. $\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta$

Checking Antiderivative Formulas

Verify the formulas in Exercises 71–82 by differentiation.

71. $\int (7x - 2)^3\, dx = \frac{(7x - 2)^4}{28} + C$

72. $\int (3x + 5)^{-2}\, dx = -\frac{(3x + 5)^{-1}}{3} + C$

73. $\int \sec^2 (5x - 1)\, dx = \frac{1}{5} \tan (5x - 1) + C$

74. $\int \csc^2\left(\frac{x - 1}{3}\right) dx = -3 \cot\left(\frac{x - 1}{3}\right) + C$

75. $\int \frac{1}{(x + 1)^2} dx = -\frac{1}{x + 1} + C$

76. $\int \frac{1}{(x + 1)^2} dx = \frac{x}{x + 1} + C$

77. $\int \frac{1}{x + 1} dx = \ln |x + 1| + C, \quad x \neq -1$

78. $\int xe^x\, dx = xe^x - e^x + C$

79. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

80. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

81. $\int \frac{\tan^{-1} x}{x^2} dx = \ln x - \frac{1}{2} \ln (1 + x^2) - \frac{\tan^{-1} x}{x} + C$

82. $\int (\sin^{-1} x)^2\, dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2} \sin^{-1} x + C$

83. Right, or wrong? Say which for each formula and give a brief reason for each answer.

 a. $\int x \sin x\, dx = \frac{x^2}{2} \sin x + C$

 b. $\int x \sin x\, dx = -x \cos x + C$

 c. $\int x \sin x\, dx = -x \cos x + \sin x + C$

84. Right, or wrong? Say which for each formula and give a brief reason for each answer.

 a. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{\sec^3 \theta}{3} + C$

 b. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \tan^2 \theta + C$

 c. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \sec^2 \theta + C$

85. Right, or wrong? Say which for each formula and give a brief reason for each answer.

 a. $\int (2x + 1)^2\, dx = \frac{(2x + 1)^3}{3} + C$

 b. $\int 3(2x + 1)^2\, dx = (2x + 1)^3 + C$

 c. $\int 6(2x + 1)^2\, dx = (2x + 1)^3 + C$

86. Right, or wrong? Say which for each formula and give a brief reason for each answer.

 a. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x + C}$

 b. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x} + C$

 c. $\int \sqrt{2x + 1}\, dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

87. Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4} dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

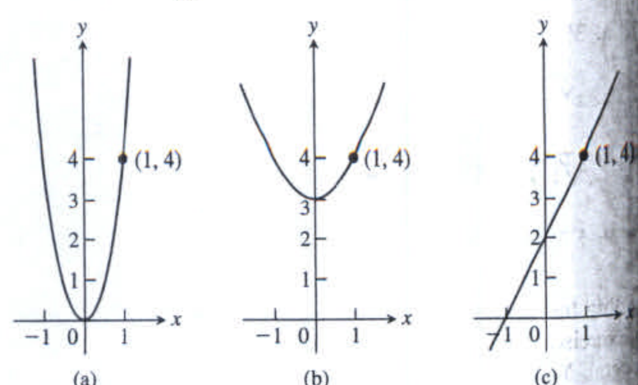
88. Right, or wrong? Give a brief reason why.

$$\int \frac{x \cos (x^2) - \sin (x^2)}{x^2} dx = \frac{\sin (x^2)}{x} + C$$

Initial Value Problems

89. Which of the following graphs shows the solution of the initial value problem

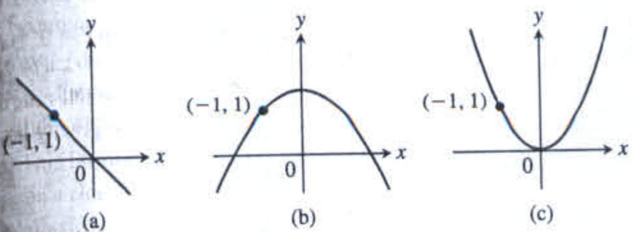
$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$



Give reasons for your answer.

90. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



Give reasons for your answer.

Solve the initial value problems in Exercises 91–112.

91. $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

92. $\frac{dy}{dx} = 10 - x, \quad y(0) = -1$

93. $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

94. $\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

95. $\frac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

96. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(4) = 0$

97. $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

98. $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

99. $\frac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

100. $\frac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

101. $\frac{dv}{dt} = \frac{1}{2} \sec t \tan t, \quad v(0) = 1$

102. $\frac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\frac{\pi}{2}\right) = -7$

103. $\frac{dv}{dt} = \frac{3}{t\sqrt{t^2 - 1}}, \quad t > 1, v(2) = 0$

104. $\frac{dv}{dt} = \frac{8}{1 + t^2} + \sec^2 t, \quad v(0) = 1$

105. $\frac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

106. $\frac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

107. $\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left.\frac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

108. $\frac{d^2s}{dt^2} = \frac{3t}{8}; \quad \left.\frac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

109. $\frac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

110. $\frac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\frac{1}{2}, \quad \theta(0) = \sqrt{2}$

111. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

112. $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

113. Find the curve $y = f(x)$ in the xy-plane that passes through the point (9, 4) and whose slope at each point is $3\sqrt{x}$.

114. a. Find a curve $y = f(x)$ with the following properties:

i) $\frac{d^2y}{dx^2} = 6x$

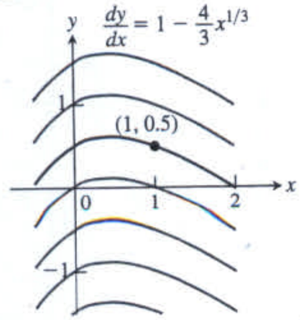
ii) Its graph passes through the point (0, 1) and has a horizontal tangent there.

b. How many curves like this are there? How do you know?

Solution (Integral) Curves

Exercises 115–118 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

115.



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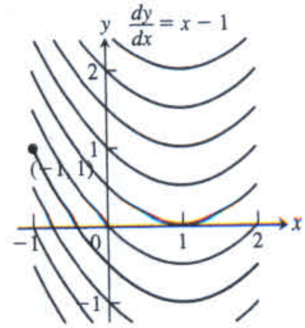

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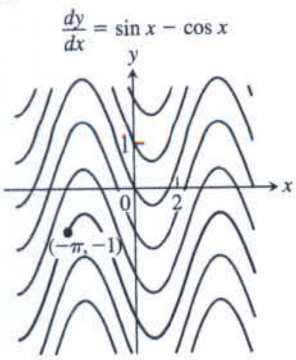

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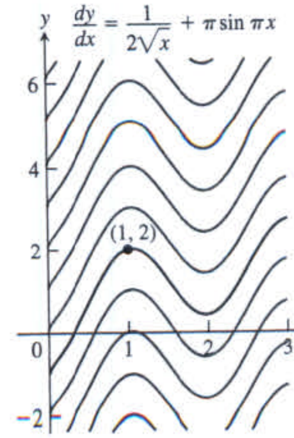


Applications

119. Finding displacement from an antiderivative of velocity

a. Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

i) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

ii) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

iii) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

b. Suppose that the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

120. Liftoff from Earth A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later?

121. Stopping a car in time You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

1. Solve the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = -k$ (k constant)

Initial conditions: $\dfrac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

Measuring time and distance from when the brakes are applied

2. Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

3. Find the value of k that makes $s = 242$ for the value of t you found in Step 2.

122. Stopping a motorcycle The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

123. Motion along a coordinate line A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - \left(3/\sqrt{t}\right)$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

a. the velocity $v = ds/dt$ in terms of t

b. the position s in terms of t.

T **124. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

Differential equation: $\dfrac{d^2s}{dt^2} = -5.2 \text{ ft/sec}^2$

Initial conditions: $\dfrac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

125. Motion with constant acceleration The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \qquad (1)$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = a$

Initial conditions: $\dfrac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

126. Free fall near the surface of a planet For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of g length-units/sec^2, Equation (1) in Exercise 125 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad (2)$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 125, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

127. Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a. $\displaystyle\int f(x)\,dx$

b. $\displaystyle\int g(x)\,dx$

c. $\displaystyle\int [-f(x)]\,dx$

d. $\displaystyle\int [-g(x)]\,dx$

e. $\displaystyle\int [f(x) + g(x)]\,dx$

f. $\displaystyle\int [f(x) - g(x)]\,dx$

128. Uniqueness of solutions If differentiable functions $y = F(x)$ and $y = g(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

COMPUTER EXPLORATIONS

Use a CAS to solve the initial value problems in Exercises 129–132. Plot the solution curves.

129. $y' = \cos^2 x + \sin x, \quad y(\pi) = 1$

130. $y' = \frac{1}{x} + x, \quad y(1) = -1$

131. $y' = \dfrac{1}{\sqrt{4 - x^2}}, \quad y(0) = 2$

132. $y'' = \frac{2}{x} + \sqrt{x}, \quad y(1) = 0, \quad y'(1) = 0$

The choices for the c_k could maximize or minimize the value of f in the kth subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. The finite sum approximations we looked at improved as we took more subintervals of thinner width.

Exercises 5.1

Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

a. a lower sum with two rectangles of equal width.

b. a lower sum with four rectangles of equal width.

c. an upper sum with two rectangles of equal width.

d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$.

2. $f(x) = x^3$ between $x = 0$ and $x = 1$.

3. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

4. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.

6. $f(x) = x^3$ between $x = 0$ and $x = 1$.

7. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

8. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. Distance traveled The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

10. Distance traveled upstream You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

a. left-endpoint values.

b. right-endpoint values.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

11. Length of a road You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road using

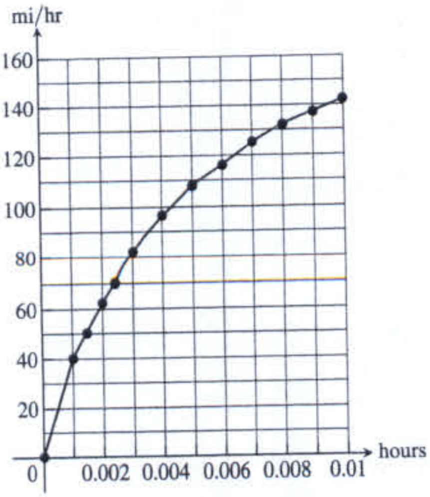
a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

12. Distance from velocity data The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		



a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

13. Free fall with air resistance An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown:

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

a. Find an upper estimate for the speed when $t = 5$.

b. Find a lower estimate for the speed when $t = 5$.

c. Find an upper estimate for the distance fallen when $t = 3$.

14. Distance traveled by a projectile An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

a. Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32\ \text{ft}/\text{sec}^2$ for the gravitational acceleration.

b. Find a lower estimate for the height attained after 5 sec.

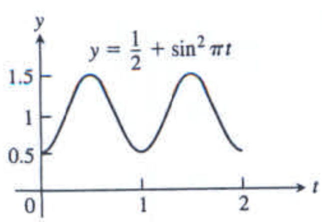
Average Value of a Function

In Exercises 15–18, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

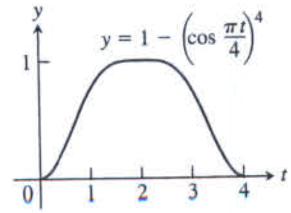
15. $f(x) = x^3$ on $[0, 2]$

16. $f(x) = 1/x$ on $[1, 9]$

17. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



18. $f(t) = 1 - \left(\cos \dfrac{\pi t}{4}\right)^4$ on $[0, 4]$



Examples of Estimations

19. Water pollution Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4
Leakage (gal/h)	50	70	97	136	190

Time (h)	5	6	7	8
Leakage (gal/h)	265	369	516	720

a. Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

b. Repeat part (a) for the quantity of oil that has escaped after 8 hours.

c. The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

20. Air pollution A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**
Pollutant release rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Pollutant release rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

a. Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day to be released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

b. In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

21. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

    a. 4 (square) b. 8 (octagon) c. 16

    d. Compare the areas in parts (a), (b), and (c) with the area of the circle.

22. *(Continuation of Exercise 21.)*

    a. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

    b. Compute the limit of the area of the inscribed polygon as $n \to \infty$.

    c. Repeat the computations in parts (a) and (b) for a circle of radius r.

COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

a. Plot the functions over the given interval.

b. Subdivide the interval into $n = 100$, 200, and 1000 subintervals of equal length and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) = $ (average value) for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

23. $f(x) = \sin x$ on $[0, \pi]$ **24.** $f(x) = \sin^2 x$ on $[0, \pi]$

25. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$ **26.** $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

5.2 Sigma Notation and Limits of Finite Sums

In estimating with finite sums in Section 5.1, we encountered sums with many terms (up to 1000 in Table 5.1, for instance). In this section we introduce a more convenient notation for sums with a large number of terms. After describing the notation and stating several of its properties, we look at what happens to a finite sum approximation as the number of terms approaches infinity.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n .$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.

$$\sum_{k=1}^{n} a_k$$

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

a_k is a formula for the kth term.

The index k starts at $k = 1$.

Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The lower limit of summation does not have to be 1; it can be any integer.

The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals to construct a Riemann sum, a single limiting value is approached as the subinterval widths, controlled by the norm of the partition, approach zero.

ercises 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$
2. $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$
3. $\displaystyle\sum_{k=1}^{4} \cos k\pi$
4. $\displaystyle\sum_{k=1}^{5} \sin k\pi$
5. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$
6. $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$
7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?
   a. $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ b. $\displaystyle\sum_{k=0}^{5} 2^{k}$ c. $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$
8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?
   a. $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ b. $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ c. $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$
9. Which formula is not equivalent to the other two?
   a. $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ b. $\displaystyle\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ c. $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$
10. Which formula is not equivalent to the other two?
   a. $\displaystyle\sum_{k=1}^{4} (k-1)^2$ b. $\displaystyle\sum_{k=-1}^{3} (k+1)^2$ c. $\displaystyle\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

11. $1 + 2 + 3 + 4 + 5 + 6$
12. $1 + 4 + 9 + 16$
13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
14. $2 + 4 + 6 + 8 + 10$
15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$
16. $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\displaystyle\sum_{k=1}^{n} a_k = -5$ and $\displaystyle\sum_{k=1}^{n} b_k = 6$. Find the values of
   a. $\displaystyle\sum_{k=1}^{n} 3a_k$ b. $\displaystyle\sum_{k=1}^{n} \frac{b_k}{6}$ c. $\displaystyle\sum_{k=1}^{n} (a_k + b_k)$
   d. $\displaystyle\sum_{k=1}^{n} (a_k - b_k)$ e. $\displaystyle\sum_{k=1}^{n} (b_k - 2a_k)$
18. Suppose that $\displaystyle\sum_{k=1}^{n} a_k = 0$ and $\displaystyle\sum_{k=1}^{n} b_k = 1$. Find the values of
   a. $\displaystyle\sum_{k=1}^{n} 8a_k$ b. $\displaystyle\sum_{k=1}^{n} 250b_k$
   c. $\displaystyle\sum_{k=1}^{n} (a_k + 1)$ d. $\displaystyle\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

19. a. $\displaystyle\sum_{k=1}^{10} k$ b. $\displaystyle\sum_{k=1}^{10} k^2$ c. $\displaystyle\sum_{k=1}^{10} k^3$
20. a. $\displaystyle\sum_{k=1}^{13} k$ b. $\displaystyle\sum_{k=1}^{13} k^2$ c. $\displaystyle\sum_{k=1}^{13} k^3$
21. $\displaystyle\sum_{k=1}^{7} (-2k)$
22. $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$
23. $\displaystyle\sum_{k=1}^{6} (3 - k^2)$
24. $\displaystyle\sum_{k=1}^{6} (k^2 - 5)$
25. $\displaystyle\sum_{k=1}^{5} k(3k + 5)$
26. $\displaystyle\sum_{k=1}^{7} k(2k + 1)$
27. $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$
28. $\displaystyle\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$
29. a. $\displaystyle\sum_{k=1}^{7} 3$ b. $\displaystyle\sum_{k=1}^{500} 7$ c. $\displaystyle\sum_{k=3}^{264} 10$
30. a. $\displaystyle\sum_{k=9}^{36} k$ b. $\displaystyle\sum_{k=3}^{17} k^2$ c. $\displaystyle\sum_{k=18}^{71} k(k - 1)$
31. a. $\displaystyle\sum_{k=1}^{n} 4$ b. $\displaystyle\sum_{k=1}^{n} c$ c. $\displaystyle\sum_{k=1}^{n} (k - 1)$
32. a. $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ b. $\displaystyle\sum_{k=1}^{n} \frac{c}{n}$ c. $\displaystyle\sum_{k=1}^{n} \frac{k}{n^2}$

Riemann Sums

In Exercises 33–36, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

33. $f(x) = x^2 - 1, \quad [0, 2]$

34. $f(x) = -x^2, \quad [0, 1]$

35. $f(x) = \sin x, \quad [-\pi, \pi]$

36. $f(x) = \sin x + 1, \quad [-\pi, \pi]$

37. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

38. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 39–46, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

39. $f(x) = 1 - x^2$ over the interval $[0, 1]$.

40. $f(x) = 2x$ over the interval $[0, 3]$.

41. $f(x) = x^2 + 1$ over the interval $[0, 3]$.

42. $f(x) = 3x^2$ over the interval $[0, 1]$.

43. $f(x) = x + x^2$ over the interval $[0, 1]$.

44. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.

45. $f(x) = 2x^3$ over the interval $[0, 1]$.

46. $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

5.3 The Definite Integral

In Section 5.2 we investigated the limit of a finite sum for a function defined over a closed interval $[a, b]$ using n subintervals of equal width (or length), $(b - a)/n$. In this section we consider the limit of more general Riemann sums as the norm of the partitions of $[a, b]$ approaches zero. For general Riemann sums, the subintervals of the partitions need not have equal widths. The limiting process then leads to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

Definition of the Definite Integral

The definition of the definite integral is based on the idea that for certain functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann sums approach a limiting value J. What we mean by this limit is that a Riemann sum will be close to the number J provided that the norm of its partition is sufficiently small (so that all of its subintervals have thin enough widths). We introduce the symbol ϵ as a small positive number that specifies how close to J the Riemann sum must be, and the symbol δ as a second small positive number that specifies how small the norm of a partition must be in order for convergence to happen. We now define this limit precisely.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

The definition involves a limiting process in which the norm of the partition goes to zero.

We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit J, no matter what choices are made. When the limit exists we write it as the definite integral

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

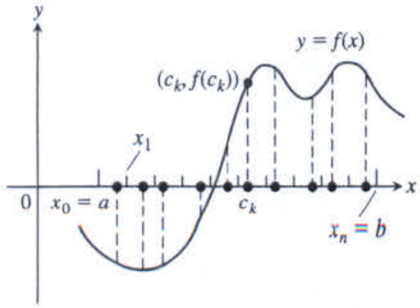


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

Alternatively, we can use the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n. A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each (Figure 5.14). The average of the n sampled values is

$$\begin{aligned}\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n}\sum_{k=1}^{n} f(c_k) \\ &= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) \qquad \Delta x = \frac{b-a}{n},\ \text{so } \frac{1}{n} = \frac{\Delta x}{b-a} \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\,\Delta x. \qquad \text{Constant Multiple Rule}\end{aligned}$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the size of the sample and let the norm of the partition approach zero, the average approaches $(1/(b - a))\int_a^b f(x)\,dx$. Both points of view lead us to the following definition.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

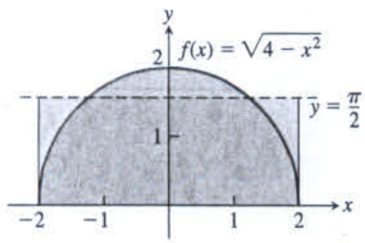


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

EXAMPLE 5 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 5.15).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2}\cdot \pi r^2 = \frac{1}{2}\cdot \pi(2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2}\sqrt{4 - x^2}\,dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4 - x^2}\,dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 5.15).

Exercises 5.3

Interpreting Limits of Sums as Integrals

Express the limits in Exercises 1–8 as definite integrals.

1. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} c_k^{\,2}\,\Delta x_k$, where P is a partition of $[0, 2]$
2. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} 2c_k^{\,3}\,\Delta x_k$, where P is a partition of $[-1, 0]$
3. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} (c_k^{\,2} - 3c_k)\,\Delta x_k$, where P is a partition of $[-7, 5]$
4. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} \left(\frac{1}{c_k}\right)\Delta x_k$, where P is a partition of $[1, 4]$
5. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} \frac{1}{1 - c_k}\,\Delta x_k$, where P is a partition of $[2, 3]$

6. $\lim_{\|P\|\to 0}\sum_{k=1}^{n}\sqrt{4-c_k^2}\,\Delta x_k$, where P is a partition of $[0, 1]$

7. $\lim_{\|P\|\to 0}\sum_{k=1}^{n}(\sec c_k)\,\Delta x_k$, where P is a partition of $[-\pi/4, 0]$

8. $\lim_{\|P\|\to 0}\sum_{k=1}^{n}(\tan c_k)\,\Delta x_k$, where P is a partition of $[0, \pi/4]$

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.6 to find

a. $\int_2^2 g(x)\,dx$ b. $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ d. $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ f. $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.6 to find

a. $\int_1^9 -2f(x)\,dx$ b. $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ d. $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ f. $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ b. $\int_1^2 \sqrt{3}f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ d. $\int_1^2 [-f(x)]\,dx$

12. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$ b. $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$ d. $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ b. $\int_4^3 f(t)\,dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\int_1^3 h(r)\,dr$ b. $-\int_3^1 h(u)\,du$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use known area formulas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$

16. $\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\int_{-3}^3 \sqrt{9 - x^2}\,dx$

18. $\int_{-4}^0 \sqrt{16 - x^2}\,dx$

19. $\int_{-2}^1 |x|\,dx$

20. $\int_{-1}^1 (1 - |x|)\,dx$

21. $\int_{-1}^1 (2 - |x|)\,dx$

22. $\int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right) dx$

Use known area formulas to evaluate the integrals in Exercises 23–28.

23. $\int_0^b \frac{x}{2}\,dx, \quad b > 0$

24. $\int_0^b 4x\,dx, \quad b > 0$

25. $\int_a^b 2s\,ds, \quad 0 < a < b$

26. $\int_a^b 3t\,dt, \quad 0 < a < b$

27. $f(x) = \sqrt{4 - x^2}$ on **a.** $[-2, 2]$, **b.** $[0, 2]$

28. $f(x) = 3x + \sqrt{1 - x^2}$ on **a.** $[-1, 0]$, **b.** $[-1, 1]$

Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 29–40.

29. $\int_1^{\sqrt{2}} x\,dx$

30. $\int_{0.5}^{2.5} x\,dx$

31. $\int_{\pi}^{2\pi} \theta\,d\theta$

32. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$

33. $\int_0^{\sqrt[3]{7}} x^2\,dx$

34. $\int_0^{0.3} s^2\,ds$

35. $\int_0^{1/2} t^2\,dt$

36. $\int_0^{\pi/2} \theta^2\,d\theta$

37. $\int_a^{2a} x\,dx$

38. $\int_a^{\sqrt{3}a} x\,dx$

39. $\int_0^{\sqrt[3]{b}} x^2\,dx$

40. $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.6 and Equations (2)–(4) to evaluate the integrals in Exercises 41–50.

41. $\int_3^1 7\,dx$

42. $\int_0^2 5x\,dx$

43. $\int_0^2 (2t - 3)\,dt$

44. $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$

46. $\int_3^0 (2z - 3)\,dz$

47. $\int_1^2 3u^2\,du$

48. $\int_{1/2}^1 24u^2\,du$

49. $\int_0^2 (3x^2 + x - 5)\,dx$

50. $\int_1^0 (3x^2 + x - 5)\,dx$

Finding Area by Definite Integrals

In Exercises 51–54, use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

51. $y = 3x^2$

52. $y = \pi x^2$

53. $y = 2x$

54. $y = \frac{x}{2} + 1$

inding Average Value

n Exercises 55–62, graph the function and find its average value over he given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

56. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$

57. $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

Definite Integrals as Limits of Sums

Use the method of Example 4a or Equation (1) to evaluate the definite integrals in Exercises 63–70.

63. $\displaystyle\int_a^b c\, dx$

64. $\displaystyle\int_0^2 (2x + 1)\, dx$

65. $\displaystyle\int_a^b x^2\, dx,\quad a < b$

66. $\displaystyle\int_{-1}^0 (x - x^2)\, dx$

67. $\displaystyle\int_{-1}^2 (3x^2 - 2x + 1)\, dx$

68. $\displaystyle\int_{-1}^1 x^3\, dx$

69. $\displaystyle\int_a^b x^3\, dx,\quad a < b$

70. $\displaystyle\int_0^1 (3x - x^3)\, dx$

Theory and Examples

71. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\, dx?$$

(*Hint:* Where is the integrand positive?)

72. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\, dx?$$

73. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2} dx.$$

74. (*Continuation of Exercise 73.*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2} dx \quad\text{and}\quad \int_{0.5}^1 \frac{1}{1 + x^2} dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2} dx.$$

75. Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

76. Show that the value of $\int_0^1 \sqrt{x + 8}\, dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

77. **Integrals of nonnegative functions** Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \ge 0.$$

78. **Integrals of nonpositive functions** Show that if f is integrable then

$$f(x) \le 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \le 0.$$

79. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\, dx$.

80. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\, dx$.

81. If $\operatorname{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\operatorname{av}(f)$ should have the same integral over $[a, b]$ as f. Does it? That is, does

$$\int_a^b \operatorname{av}(f)\, dx = \int_a^b f(x)\, dx?$$

Give reasons for your answer.

82. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\operatorname{av}(f + g) = \operatorname{av}(f) + \operatorname{av}(g)$

b. $\operatorname{av}(kf) = k \operatorname{av}(f)$ (any number k)

c. $\operatorname{av}(f) \le \operatorname{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

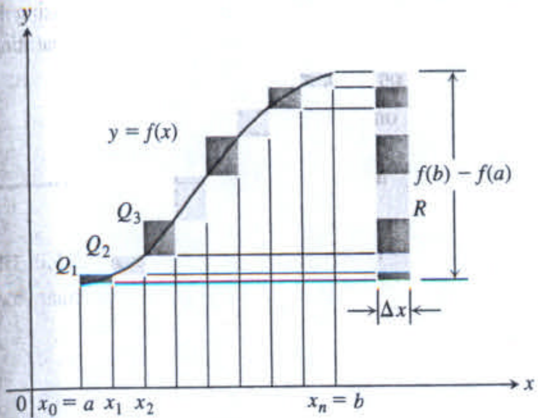
83. **Upper and lower sums for increasing functions**

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\, \Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.



84. Upper and lower sums for decreasing functions *(Continuation of Exercise 83.)*

a. Draw a figure like the one in Exercise 83 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 83a.

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \le |f(b) - f(a)|\, \Delta x_{\max}$$

of Exercise 83b still holds and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.

85. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

86. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.

a. If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n$$

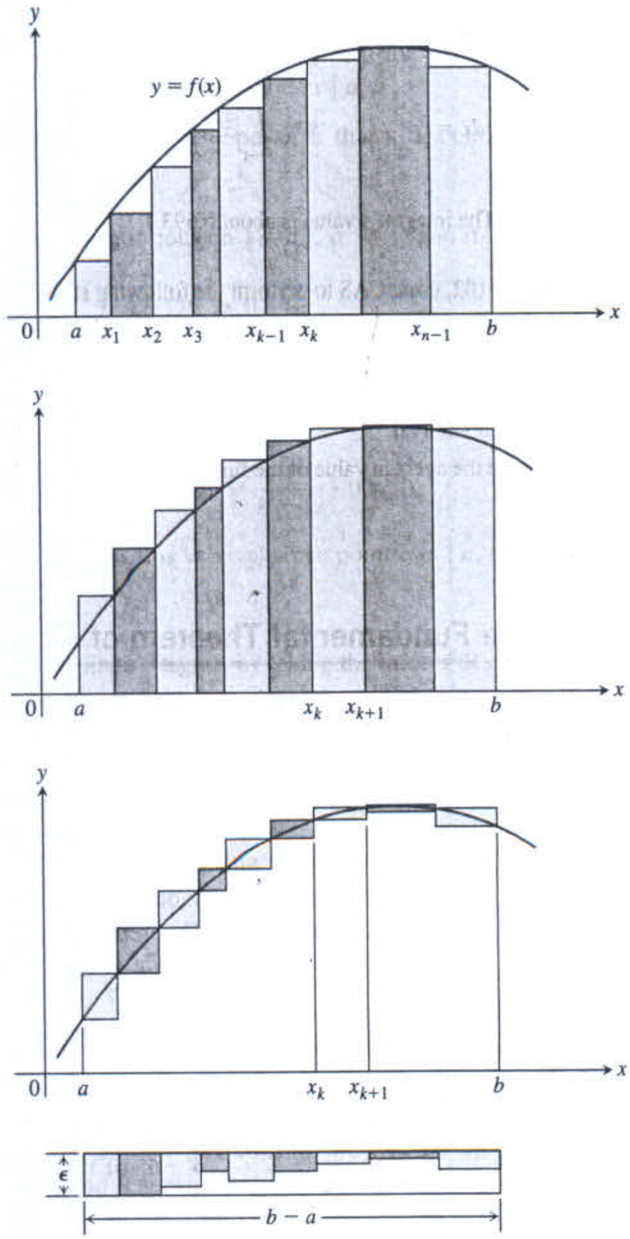
and the shaded regions in the first part of the figure.

b. If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n$$

and the shaded regions in the second part of the figure.

c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.



87. We say f is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \le \epsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.

88. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer.

COMPUTER EXPLORATIONS

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 89–94. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

89. $\int_0^1 (1 - x)\,dx = \frac{1}{2}$

90. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

91. $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = 0$

92. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = 1$

93. $\displaystyle\int_{-1}^{1} |x|\,dx = 1$

94. $\displaystyle\int_1^2 \frac{1}{x}dx$ (The integral's value is about 0.693.)

In Exercises 95–102, use a CAS to perform the following steps:

a. Plot the functions over the given interval.

b. Partition the interval into $n = 100$, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) =$ (average value) for x using the aver age value calculated in part (c) for the $n = 1000$ partitioning.

95. $f(x) = \sin x$ on $[0, \pi]$

96. $f(x) = \sin^2 x$ on $[0, \pi]$

97. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

98. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

99. $f(x) = xe^{-x}$ on $[0, 1]$

100. $f(x) = e^{-x^2}$ on $[0, 1]$

101. $f(x) = \frac{\ln x}{x}$ on $[2, 5]$

102. $f(x) = \frac{1}{\sqrt{1 - x^2}}$ on $\left[0, \frac{1}{2}\right]$

5.4 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY
Sir Isaac Newton
(1642–1727)

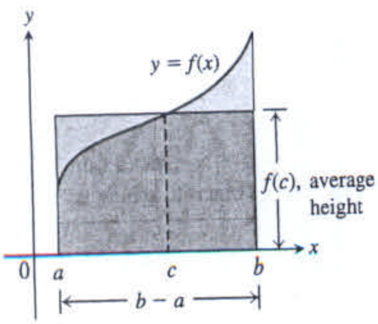


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

In this section we present the Fundamental Theorem of Calculus, which is the central theore of integral calculus. It connects integration and differentiation, enabling us to compute inte grals using an antiderivative of the integrand function rather than by taking limits of Rieman sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and starte mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we present an integral version of the Mean Value Theorem, which another important theorem of integral calculus and is used to prove the Fundament Theorem. We also find that the net change of a function over an interval is the integral its rate of change, as suggested by Example 3 in Section 5.1.

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\,dx$ divided by the length or wid $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that th average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined ov the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c the interval $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function a base width $b - a$ has exactly the same area as the region beneath the graph of f from a to

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.6, Rule 6) by $(b -$ we obtain

$$\min f \leq \frac{1}{b - a}\int_a^b f(x)\,dx \leq \max f.$$

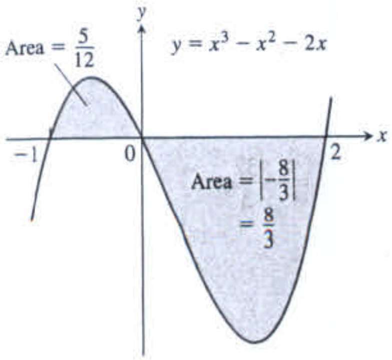


FIGURE 5.22 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x,\ -1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.22). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

Exercises 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

1. $\int_0^2 x(x - 3)\,dx$
2. $\int_{-1}^{1} (x^2 - 2x + 3)\,dx$
3. $\int_{-2}^{2} \frac{3}{(x + 3)^4}\,dx$
4. $\int_{-1}^{1} x^{299}\,dx$
5. $\int_1^4 \left(3x^2 - \frac{x^3}{4}\right) dx$
6. $\int_{-2}^{3} (x^3 - 2x + 3)\,dx$
7. $\int_0^1 \left(x^2 + \sqrt{x}\right) dx$
8. $\int_1^{32} x^{-6/5}\,dx$
9. $\int_0^{\pi/3} 2\sec^2 x\,dx$
10. $\int_0^{\pi} (1 + \cos x)\,dx$
11. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$
12. $\int_0^{\pi/3} 4\,\frac{\sin u}{\cos^2 u}\,du$
13. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\,dt$
14. $\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$
15. $\int_0^{\pi/4} \tan^2 x\,dx$
16. $\int_0^{\pi/6} (\sec x + \tan x)^2\,dx$
17. $\int_0^{\pi/8} \sin 2x\,dx$
18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$
19. $\int_1^{-1} (r + 1)^2\,dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4)\,dt$
21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$
22. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\,dy$
23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\,ds$
24. $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}}\,dx$
25. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$
26. $\int_0^{\pi/3} (\cos x + \sec x)^2\,dx$
27. $\int_{-4}^{4} |x|\,dx$
28. $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$
29. $\int_0^{\ln 2} e^{3x}\,dx$
30. $\int_1^2 \left(\frac{1}{x} - e^{-x}\right) dx$
31. $\int_0^{1/2} \frac{4}{\sqrt{1 - x^2}}\,dx$
32. $\int_0^{1/\sqrt{3}} \frac{dx}{1 + 4x^2}$
33. $\int_2^4 x^{\pi - 1}\,dx$
34. $\int_{-1}^{0} \pi^{x-1}\,dx$

In Exercises 35–38, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep in mind the Chain Rule in guessing an antiderivative. You will learn how to find such antiderivatives in the next section.)

35. $\int_0^1 xe^{x^2}\,dx$
36. $\int_1^2 \frac{\ln x}{x}\,dx$
37. $\int_2^5 \frac{x\,dx}{\sqrt{1 + x^2}}$
38. $\int_0^{\pi/3} \sin^2 x\cos x\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 39–44.

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

39. $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \cos t\, dt$

40. $\dfrac{d}{dx}\displaystyle\int_1^{\sin x} 3t^2\, dt$

41. $\dfrac{d}{dt}\displaystyle\int_0^{t^4} \sqrt{u}\, du$

42. $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan\theta} \sec^2 y\, dy$

43. $\dfrac{d}{dx}\displaystyle\int_0^{x^3} e^{-t}\, dt$

44. $\dfrac{d}{dt}\displaystyle\int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1 - x^2}}\right) dx$

Find dy/dx in Exercises 45–56.

45. $y = \displaystyle\int_0^x \sqrt{1 + t^2}\, dt$

46. $y = \displaystyle\int_1^x \frac{1}{t}\, dt, \quad x > 0$

47. $y = \displaystyle\int_{\sqrt{x}}^0 \sin(t^2)\, dt$

48. $y = x\displaystyle\int_2^{x^2} \sin(t^3)\, dt$

49. $y = \displaystyle\int_{-1}^x \frac{t^2}{t^2 + 4}\, dt - \int_3^x \frac{t^2}{t^2 + 4}\, dt$

50. $y = \left(\displaystyle\int_0^x (t^3 + 1)^{10}\, dt\right)^3$

51. $y = \displaystyle\int_0^{\sin x} \frac{dt}{\sqrt{1 - t^2}}, \quad |x| < \frac{\pi}{2}$

52. $y = \displaystyle\int_{\tan x}^0 \frac{dt}{1 + t^2}$

53. $y = \displaystyle\int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\, dt$

54. $y = \displaystyle\int_{2^x}^1 \sqrt[3]{t}\, dt$

55. $y = \displaystyle\int_0^{\sin^{-1} x} \cos t\, dt$

56. $y = \displaystyle\int_{-1}^{x^{1/\pi}} \sin^{-1} t\, dt$

Area

In Exercises 57–60, find the total area between the region and the x-axis.

57. $y = -x^2 - 2x, \quad -3 \le x \le 2$

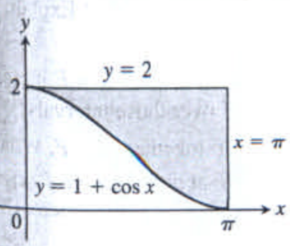
58. $y = 3x^2 - 3, \quad -2 \le x \le 2$

59. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

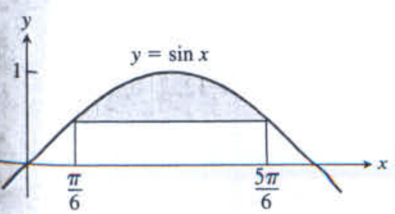
60. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 61–64.

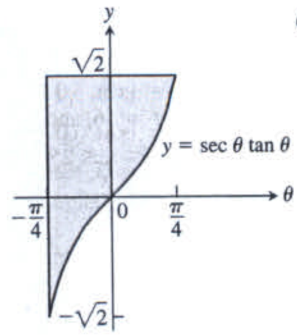
61.



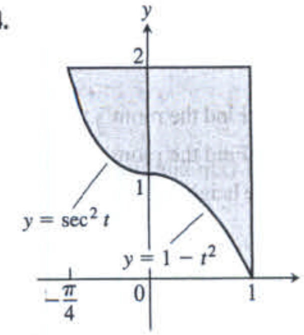
62.



63.



64.



Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 65–68. Which function solves which problem? Give brief reasons for your answers.

a. $y = \displaystyle\int_1^x \frac{1}{t}\, dt - 3$

b. $y = \displaystyle\int_0^x \sec t\, dt + 4$

c. $y = \displaystyle\int_{-1}^x \sec t\, dt + 4$

d. $y = \displaystyle\int_\pi^x \frac{1}{t}\, dt - 3$

65. $\dfrac{dy}{dx} = \dfrac{1}{x}, \quad y(\pi) = -3$

66. $y' = \sec x, \quad y(-1) = 4$

67. $y' = \sec x, \quad y(0) = 4$

68. $y' = \dfrac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 69 and 70 in terms of integrals.

69. $\dfrac{dy}{dx} = \sec x, \quad y(2) = 3$

70. $\dfrac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

Theory and Examples

71. **Archimedes' area formula for parabolic arches** Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

72. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

73. **Cost from marginal cost** The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

74. **Revenue from marginal revenue** Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x + 1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

75. The temperature $T(°F)$ of a room at time t minutes is given by

$$T = 85 - 3\sqrt{25 - t} \quad \text{for} \quad 0 \le t \le 25.$$

a. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

b. Find the room's average temperature for $0 \le t \le 25$.

76. The height H(ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \text{for} \quad 0 \le t \le 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \le t \le 8$.

77. Suppose that $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

78. Find $f(4)$ if $\int_0^x f(t)\,dt = x \cos \pi x$.

79. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1 + t}\,dt$$

at $x = 1$.

80. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t - 1)\,dt$$

at $x = -1$.

81. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of dg/dx crosses the x-axis at $x = 1$.

82. Another proof of the Evaluation Theorem

a. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let F be any antiderivative of f. Show that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

b. Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some c_i in the interval (x_{i-1}, x_i). Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.

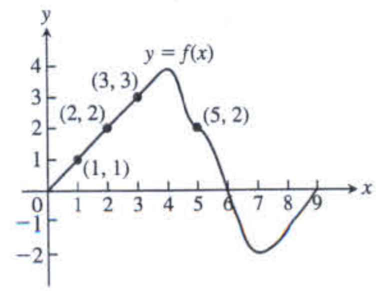
c. From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

83. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

f. When is the particle moving toward the origin? Away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

84. Find $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} \int_1^x \frac{dt}{\sqrt{t}}$.

COMPUTER EXPLORATIONS

In Exercises 85–88, let $F(x) = \int_a^x f(t)\,dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a. Plot the functions f and F together over $[a, b]$.

b. Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.

c. Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d. Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

85. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

86. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \quad \left[0, \frac{9}{2}\right]$

87. $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

88. $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 89–92, let $F(x) = \int_a^{u(x)} f(t)\,dt$ for the specified a, u, and f. Use a CAS to perform the following steps and answer the questions posed.

a. Find the domain of F.

b. Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? Decreasing?

c. Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F.

d. Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

89. $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

90. $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

91. $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

92. $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

In Exercises 93 and 94, assume that f is continuous and $u(x)$ is twice-differentiable.

93. Calculate $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

94. Calculate $\dfrac{d^2}{dx^2}\displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

5.5 Indefinite Integrals and the Substitution Method

The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function. In Section 4.8 we defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f, symbolized by $\int f(x)\,dx$. Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant. The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation:

$$\int_a^b f(x)\,dx = F(b) - F(a) = \left[F(x) + C\right]_a^b = \left[\int f(x)\,dx\right]_a^b.$$

When finding the indefinite integral of a function f, remember that it always includes an arbitrary constant C.

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*. An indefinite integral $\int f(x)\,dx$ is a *function* plus an arbitrary constant C.

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions we can't easily recognize as a derivative.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n + 1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C. \tag{1}$$

Solution 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

Solution 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2 + 1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$ ■

Exercises 5.5

Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx, \quad u = 2x + 4$
2. $\int 7\sqrt{7x-1}\,dx, \quad u = 7x - 1$
3. $\int 2x(x^2+5)^{-4}\,dx, \quad u = x^2 + 5$
4. $\int \frac{4x^3}{(x^4+1)^2}\,dx, \quad u = x^4 + 1$
5. $\int (3x+2)(3x^2+4x)^4\,dx, \quad u = 3x^2 + 4x$
6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx, \quad u = 1 + \sqrt{x}$
7. $\int \sin 3x\,dx, \quad u = 3x$
8. $\int x\sin(2x^2)\,dx, \quad u = 2x^2$
9. $\int \sec 2t \tan 2t\,dt, \quad u = 2t$
10. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$
11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$
12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4 + 4y^2 + 1$
13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2} - 1$
14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$
15. $\int \csc^2 2\theta \cot 2\theta\,d\theta$
    a. Using $u = \cot 2\theta$
    b. Using $u = \csc 2\theta$
16. $\int \frac{dx}{\sqrt{5x+8}}$
    a. Using $u = 5x + 8$
    b. Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–66.

17. $\int \sqrt{3-2s}\,ds$
18. $\int \frac{1}{\sqrt{5s+4}}\,ds$
19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$
20. $\int 3y\sqrt{7-3y^2}\,dy$
21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$
22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x+2)\,dx$

24. $\int \tan^2 x \sec^2 x\,dx$

25. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\,dx$

26. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

28. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2} \sin(x^{3/2}+1)\,dx$

30. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right) dv$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

32. $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right) dt$

34. $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right) dt$

35. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1+x}}\,dx$

38. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

43. $\int x(x-1)^{10}\,dx$

44. $\int x\sqrt{4-x}\,dx$

45. $\int (x+1)^2(1-x)^5\,dx$

46. $\int (x+5)(x-5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

49. $\int \frac{x}{(x^2-4)^3}\,dx$

50. $\int \frac{x}{(2x-1)^{2/3}}\,dx$

51. $\int (\cos x)\, e^{\sin x}\,dx$

52. $\int (\sin 2\theta)\, e^{\sin^2\theta}\,d\theta$

53. $\int \frac{1}{\sqrt{x e^{-\sqrt{x}}}}\sec^2(e^{\sqrt{x}}+1)\,dx$

54. $\int \frac{1}{x^2} e^{1/x}\sec(1+e^{1/x})\tan(1+e^{1/x})\,dx$

55. $\int \frac{dx}{x\ln x}$

56. $\int \frac{\ln\sqrt{t}}{t}\,dt$

57. $\int \frac{dz}{1+e^z}$

58. $\int \frac{dx}{x\sqrt{x^4-1}}$

59. $\int \frac{5}{9+4r^2}\,dr$

60. $\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta$

61. $\int \frac{e^{\sin^{-1}x}\,dx}{\sqrt{1-x^2}}$

62. $\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}}$

63. $\int \frac{(\sin^{-1}x)^2\,dx}{\sqrt{1-x^2}}$

64. $\int \frac{\sqrt{\tan^{-1}x}\,dx}{1+x^2}$

65. $\int \frac{dy}{(\tan^{-1}y)(1+y^2)}$

66. $\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}}$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 67 and 68.

67. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

   a. $u=\tan x$, followed by $v=u^3$, then by $w=2+v$
   b. $u=\tan^3 x$, followed by $v=2+u$
   c. $u=2+\tan^3 x$

68. $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

   a. $u=x-1$, followed by $v=\sin u$, then by $w=1+v^2$
   b. $u=\sin(x-1)$, followed by $v=1+u^2$
   c. $u=1+\sin^2(x-1)$

Evaluate the integrals in Exercises 69 and 70.

69. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

70. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

71. Find the integral of $\cot x$ using a substitution like that in Example 7c.

72. Find the integral of $\csc x$ by multiplying by an appropriate form equal to 1, as in Example 8b.

Initial Value Problems

Solve the initial value problems in Exercises 73–78.

73. $\frac{ds}{dt}=12t(3t^2-1)^3,\quad s(1)=3$

74. $\frac{dy}{dx}=4x(x^2+8)^{-1/3},\quad y(0)=0$

75. $\frac{ds}{dt}=8\sin^2\left(t+\frac{\pi}{12}\right),\quad s(0)=8$

76. $\frac{dr}{d\theta}=3\cos^2\left(\frac{\pi}{4}-\theta\right),\quad r(0)=\frac{\pi}{8}$

77. $\frac{d^2s}{dt^2}=-4\sin\left(2t-\frac{\pi}{2}\right),\quad s'(0)=100,\quad s(0)=0$

78. $\frac{d^2y}{dx^2}=4\sec^2 2x\tan 2x,\quad y'(0)=4,\quad y(0)=-1$

79. The velocity of a particle moving back and forth on a line is $v=ds/dt=6\sin 2t$ m/sec for all t. If $s=0$ when $t=0$, find the value of s when $t=\pi/2$ sec.

80. The acceleration of a particle moving back and forth on a line is $a=d^2s/dt^2=\pi^2\cos\pi t$ m/sec² for all t. If $s=0$ and $v=8$ m/sec when $t=0$, find s when $t=1$ sec.

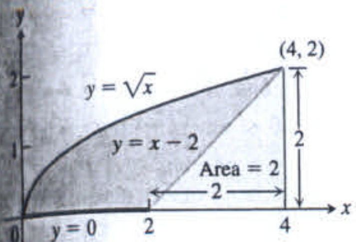


FIGURE 5.31 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Although it was easier to find the area in Example 6 by integrating with respect to y rather than x (just as we did in Example 7), there is an easier way yet. Looking at Figure 5.31, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

Exercises 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–46.

1. a. $\int_0^3 \sqrt{y+1}\,dy$ b. $\int_{-1}^0 \sqrt{y+1}\,dy$
2. a. $\int_0^1 r\sqrt{1-r^2}\,dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\,dr$
3. a. $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ b. $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$
4. a. $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$
5. a. $\int_0^1 t^3(1+t^4)^3\,dt$ b. $\int_{-1}^1 t^3(1+t^4)^3\,dt$
6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ b. $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$
7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ b. $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$
8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$
9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$
10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ b. $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$
11. a. $\int_0^1 t\sqrt{4+5t}\,dt$ b. $\int_1^9 t\sqrt{4+5t}\,dt$
12. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$ b. $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$
13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$
14. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$ b. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$
15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$
16. $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$
17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$
18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$
19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$
20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$
21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$
22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$
23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$
24. $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$
25. $\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\,d\theta$
26. $\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\,d\theta$
27. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$
28. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$
29. $\int_1^2 \frac{2\ln x}{x}\,dx$
30. $\int_2^4 \frac{dx}{x\ln x}$
31. $\int_2^4 \frac{dx}{x(\ln x)^2}$
32. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$
33. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$
34. $\int_{\pi/4}^{\pi/2} \cot t\,dt$
35. $\int_0^{\pi/3} \tan^2\theta\cos\theta\,d\theta$
36. $\int_0^{\pi/12} 6\tan 3x\,dx$

37. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1 + (\sin\theta)^2}$

38. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1 + (\cot x)^2}$

39. $\displaystyle\int_{0}^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}}$

40. $\displaystyle\int_{1}^{e^{\pi/4}} \frac{4\, dt}{t(1 + \ln^2 t)}$

41. $\displaystyle\int_{0}^{1} \frac{4\, ds}{\sqrt{4 - s^2}}$

42. $\displaystyle\int_{0}^{\sqrt[3]{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

43. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

44. $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

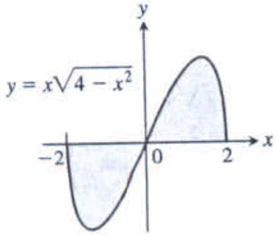
45. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

46. $\displaystyle\int_{0}^{3} \frac{y\, dy}{\sqrt{5y + 1}}$

Area

Find the total areas of the shaded regions in Exercises 47–62.

47.



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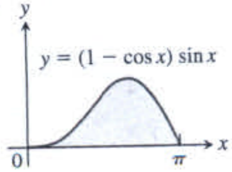

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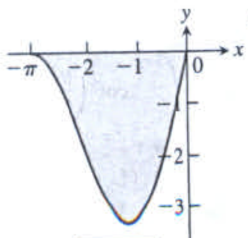
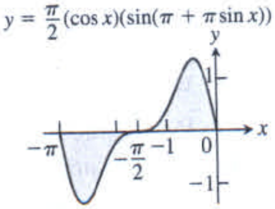


50. $y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi \sin x))$



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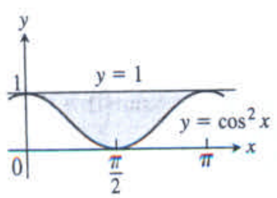

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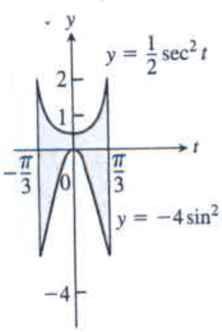

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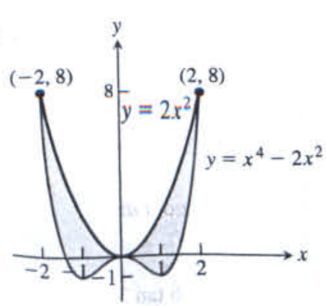

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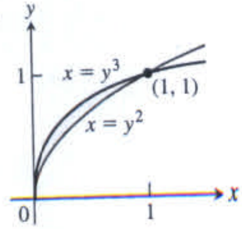

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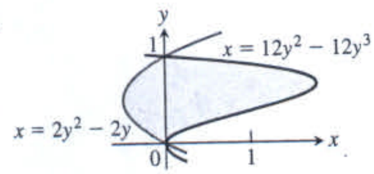

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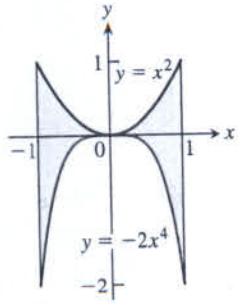

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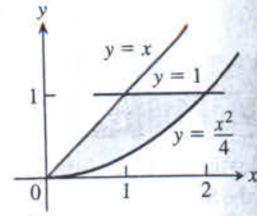

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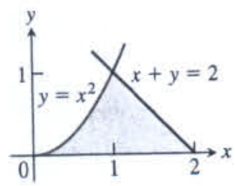

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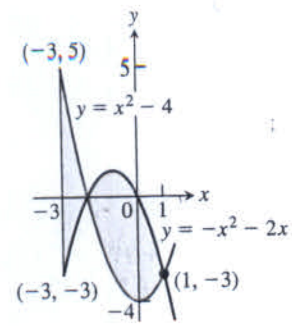

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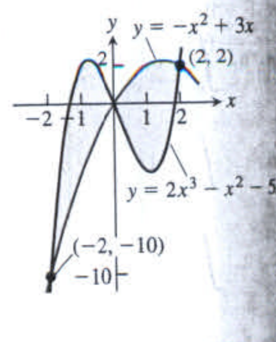
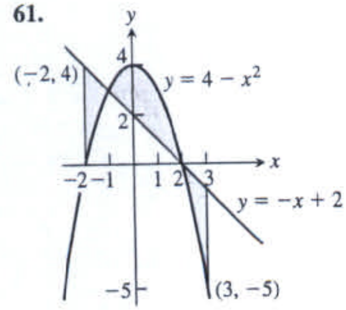
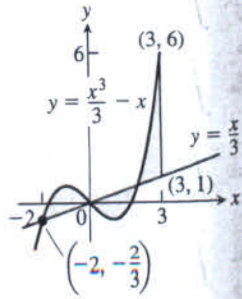


61.



62.



Find the areas of the regions enclosed by the lines and curves in Exercises 63–72.

63. $y = x^2 - 2$ and $y = 2$

64. $y = 2x - x^2$ and $y = -3$

65. $y = x^4$ and $y = 8x$

66. $y = x^2 - 2x$ and $y = x$

67. $y =$

68. $y =$

69. $y =$

70. $y =$

71. $y =$
are

72. $y =$

Find the
Exercise

73. $x =$

74. $x =$

75. y^2

76. $x -$

77. $x +$

78. $x -$

79. $x =$

80. $x =$

Find the

81. $4x^2$

82. x^3

83. $x +$

84. $x +$

Find the
cises 85

85. $y =$

86. $y =$

87. $y =$

88. $y =$

89. $y =$

90. $x =$

91. $x =$

92. $y =$

Area Be

93. Fin
cur

94. Fin
cur

95. Fin
line

96. Fin
bou
$y =$

97. Fin
$x =$

98. Fin
$x =$

99. Fin
bou
and

67. $y = x^2$ and $y = -x^2 + 4x$

68. $y = 7 - 2x^2$ and $y = x^2 + 4$

69. $y = x^4 - 4x^2 + 4$ and $y = x^2$

70. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

71. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

72. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 73–80.

73. $x = 2y^2$, $x = 0$, and $y = 3$

74. $x = y^2$ and $x = y + 2$

75. $y^2 - 4x = 4$ and $4x - y = 16$

76. $x - y^2 = 0$ and $x + 2y^2 = 3$

77. $x + y^2 = 0$ and $x + 3y^2 = 2$

78. $x - y^{2/3} = 0$ and $x + y^4 = 2$

79. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

80. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 81–84.

81. $4x^2 + y = 4$ and $x^4 - y = 1$

82. $x^3 - y = 0$ and $3x^2 - y = 4$

83. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

84. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 85–92.

85. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

86. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

87. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

88. $y = \sin(\pi x/2)$ and $y = x$

89. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

90. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

91. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

92. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

Area Between Curves

93. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

94. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

95. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

96. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

97. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

98. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

99. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

100. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

101. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \leq x \leq 2$ of the x-axis.

102. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \leq x \leq 1$ of the x-axis.

103. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

   a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

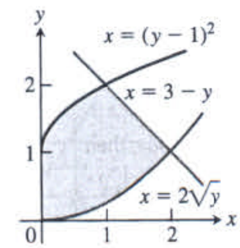
   b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

   c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

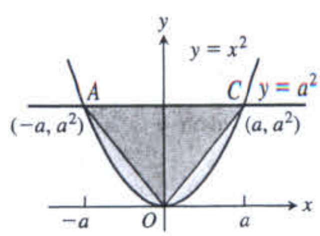
104. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

105. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

106. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



107. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

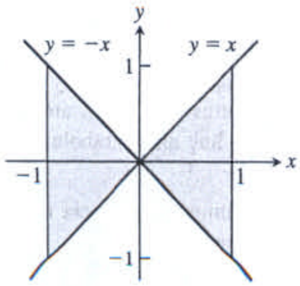


108. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

109. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\int_{-1}^{1}(x-(-x))\,dx = \int_{-1}^{1}2x\,dx$

b. $\int_{-1}^{1}(-x-(x))\,dx = \int_{-1}^{1}-2x\,dx$



110. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y=f(x)$ and $y=g(x)$ and the vertical lines $x=a$ and $x=b$ $(a<b)$ is

$$\int_a^b [f(x)-g(x)]\,dx.$$

Give reasons for your answer.

Theory and Examples

111. Suppose that $F(x)$ is an antiderivative of $f(x)=(\sin x)/x$, $x>0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\,dx$$

in terms of F.

112. Show that if f is continuous, then

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

113. Suppose that

$$\int_0^1 f(x)\,dx = 3.$$

Find

$$\int_{-1}^0 f(x)\,dx$$

if **a.** f is odd, **b.** f is even.

114. a. Show that if f is odd on $[-a,a]$, then

$$\int_{-a}^a f(x)\,dx = 0.$$

b. Test the result in part (a) with $f(x)=\sin x$ and $a=\pi/2$.

115. If f is a continuous function, find the value of the integral

$$I=\int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)}$$

by making the substitution $u=a-x$ and adding the resulting integral to I.

116. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy}\frac{1}{t}dt = \int_1^y \frac{1}{t}dt.$$

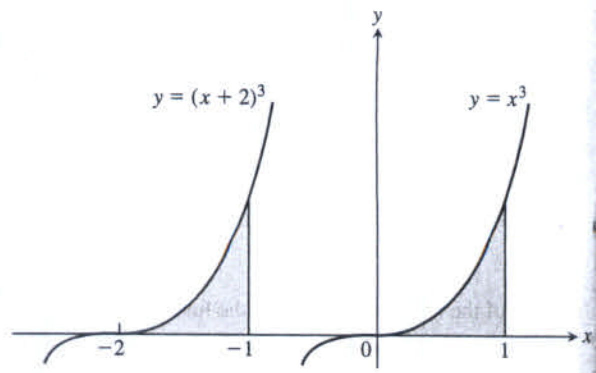
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx. \qquad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1}(x+2)^3\,dx = \int_0^1 x^3\,dx$$

because the areas of the shaded regions are congruent.



117. Use a substitution to verify Equation (1).

118. For each of the following functions, graph $f(x)$ over $[a,b]$ and $f(x+c)$ over $[a-c,b-c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x)=x^2,\quad a=0,\quad b=1,\quad c=1$

b. $f(x)=\sin x,\quad a=0,\quad b=\pi,\quad c=\pi/2$

c. $f(x)=\sqrt{x-4},\quad a=4,\quad b=8,\quad c=5$

COMPUTER EXPLORATIONS

In Exercises 119–122, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a. Plot the curves together to see what they look like and how many points of intersection they have.

b. Use the numerical equation solver in your CAS to find all the points of intersection.

c. Integrate $|f(x)-g(x)|$ over consecutive pairs of intersection values.

d. Sum together the integrals found in part (c).

119. $f(x)=\frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{1}{3},\quad g(x)=x-1$

120. $f(x)=\frac{x^4}{2}-3x^3+10,\quad g(x)=8-12x$

121. $f(x)=x+\sin(2x),\quad g(x)=x^3$

122. $f(x)=x^2\cos x,\quad g(x)=x^3-x$