Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers such as e and π that are not algebraic are called **transcendental**.

We call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

(b)
$$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$
$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x,\quad du = \frac{1}{x}dx$$
$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

Summary

In this section we used calculus to give precise definitions of the logarithmic and exponential functions. This approach is somewhat different from our earlier treatments of the polynomial, rational, and trigonometric functions. There we first defined the function and then we studied its derivatives and integrals. Here we started with an integral from which the functions of interest were obtained. The motivation behind this approach was to address mathematical difficulties that arise when we attempt to define functions such as a^x for any real number x, rational or irrational. Defining $\ln x$ as the integral of the function $1/t$ from $t = 1$ to $t = x$ enabled us to define all of the exponential and logarithmic functions, and then derive their key algebraic and analytic properties.

Exercises 7.1

Integration

Evaluate the integrals in Exercises 1–46.

1. $\int_{-3}^{-2} \frac{dx}{x}$
2. $\int_{-1}^{0} \frac{3\,dx}{3x-2}$
3. $\int \frac{2y\,dy}{y^2-25}$
4. $\int \frac{8r\,dr}{4r^2-5}$
5. $\int \frac{3\sec^2 t}{6+3\tan t}dt$
6. $\int \frac{\sec y \tan y}{2+\sec y}dy$
7. $\int \frac{dx}{2\sqrt{x}+2x}$
8. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$
9. $\int_{\ln 2}^{\ln 3} e^x\,dx$
10. $\int 8e^{(x+1)}\,dx$
11. $\int_1^4 \frac{(\ln x)^3}{2x}dx$
12. $\int \frac{\ln(\ln x)}{x\ln x}dx$
13. $\int_{\ln 4}^{\ln 9} e^{x/2}\,dx$
14. $\int \tan x \ln(\cos x)\,dx$
15. $\int \frac{e^{\sqrt{r}}}{\sqrt{r}}dr$
16. $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr$
17. $\int 2t\,e^{-t^2}\,dt$
18. $\int \frac{\ln x\,dx}{x\sqrt{\ln^2 x + 1}}$
19. $\int \frac{e^{1/x}}{x^2}dx$
20. $\int \frac{e^{-1/x^2}}{x^3}dx$
21. $\int e^{\sec \pi t}\sec \pi t \tan \pi t\,dt$
22. $\int e^{\csc(\pi+t)}\csc(\pi+t)\cot(\pi+t)\,dt$
23. $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\,dv$
24. $\int_0^{\sqrt{\ln \pi}} 2xe^{x^2}\cos(e^{x^2})\,dx$
25. $\int \frac{e^r}{1+e^r}dr$
26. $\int \frac{dx}{1+e^x}$
27. $\int_0^1 2^{-\theta}\,d\theta$
28. $\int_{-2}^{0} 5^{-\theta}\,d\theta$
29. $\int_1^{\sqrt{2}} x2^{(x^2)}\,dx$
30. $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}dx$
31. $\int_0^{\pi/2} 7^{\cos t}\sin t\,dt$
32. $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$
33. $\int_2^4 x^{2x}(1+\ln x)\,dx$
34. $\int_1^2 \frac{2^{\ln x}}{x}dx$
35. $\int_0^3 (\sqrt{2}+1)x^{\sqrt{2}}\,dx$
36. $\int_1^e x^{(\ln 2)-1}\,dx$
37. $\int \frac{\log_{10} x}{x}dx$
38. $\int_1^4 \frac{\log_2 x}{x}dx$
39. $\int_1^4 \frac{\ln 2 \log_2 x}{x}dx$
40. $\int_1^e \frac{2\ln 10 \log_{10} x}{x}dx$
41. $\int_0^2 \frac{\log_2(x+2)}{x+2}dx$
42. $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}dx$
43. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}dx$
44. $\int_2^3 \frac{2\log_2(x-1)}{x-1}dx$

45. $\int \frac{dx}{x \log_{10} x}$

46. $\int \frac{dx}{x(\log_8 x)^2}$

Initial Value Problems

Solve the initial value problems in Exercises 47–52.

47. $\frac{dy}{dt} = e^t \sin(e^t - 2), \quad y(\ln 2) = 0$

48. $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}), \quad y(\ln 4) = 2/\pi$

49. $\frac{d^2y}{dx^2} = 2e^{-x}, \quad y(0) = 1$ and $y'(0) = 0$

50. $\frac{d^2y}{dt^2} = 1 - e^{2t}, \quad y(1) = -1$ and $y'(1) = 0$

51. $\frac{dy}{dx} = 1 + \frac{1}{x}, \quad y(1) = 3$

52. $\frac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0$ and $y'(0) = 1$

Theory and Applications

53. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

54. In Section 6.2, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3 + 9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 6.2, Exercise 6, for a graph.)

Find the lengths of the curves in Exercises 55 and 56.

55. $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

56. $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

T 57. **The linearization of $\ln(1 + x)$ at $x = 0$** Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1 + x)$ near $x = 0$. We get a simpler formula this way.

a. Derive the linearization $\ln(1 + x) \approx x$ at $x = 0$.

b. Estimate to five decimal places the error involved in replacing $\ln(1 + x)$ by x on the interval $[0, 0.1]$.

c. Graph $\ln(1 + x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1 + x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

58. **The linearization of e^x at $x = 0$**

a. Derive the linear approximation $e^x \approx 1 + x$ at $x = 0$.

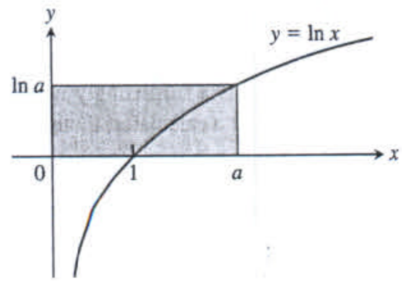
T b. Estimate to five decimal places the magnitude of the error involved in replacing e^x by $1 + x$ on the interval $[0, 0.2]$.

T c. Graph e^x and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^x? Underestimate e^x?

59. Show that for any number $a > 1$

$$\int_1^a \ln x \, dx + \int_0^{\ln a} e^y \, dy = a \ln a,$$

as suggested by the accompanying figure.

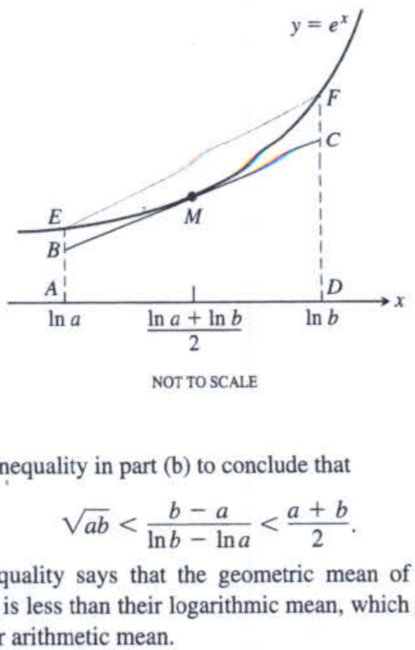


60. **The geometric, logarithmic, and arithmetic mean inequality**

a. Show that the graph of e^x is concave up over every interval of x-values.

b. Show, by reference to the accompanying figure, that if $0 < a < b$ then

$$e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a).$$



NOT TO SCALE

c. Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

Grapher Explorations

61. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

62. Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22, -2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

63. a. Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2$, 4, 8, 20, and 50 together for $0 \le x \le 23$.

b. Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

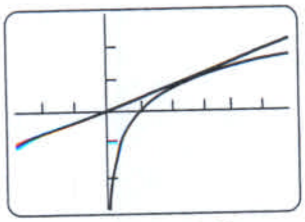
64. Does the graph of $y = \sqrt{x} - \ln x, x > 0$, have an inflection point? Try to answer the question **(a)** by graphing, **(b)** by using calculus.

T 65. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

T 66. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for some $x > 0$? Graph the two functions and explain what you see.

T 67. **Which is bigger, π^e or e^π?** Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though.

a. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.



$[-3, 6]$ by $[-3, 3]$

b. Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

c. Show that $\ln(x^e) < x$ for all positive $x \neq e$.

d. Conclude that $x^e < e^x$ for all positive $x \neq e$.

e. So which is bigger, π^e or e^π?

T 68. **A decimal representation of e** Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$ using Newton's method in Section 4.7.

Calculations with Other Bases

T 69. Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the equation $\log_a x = (\ln x)/(\ln a)$.

Find the following logarithms to five decimal places.

a. $\log_3 8$

b. $\log_7 0.5$

c. $\log_{20} 17$

d. $\log_{0.5} 7$

e. $\ln x$, given that $\log_{10} x = 2.3$

f. $\ln x$, given that $\log_2 x = 1.4$

g. $\ln x$, given that $\log_2 x = -1.5$

h. $\ln x$, given that $\log_{10} x = -0.7$

70. **Conversion factors**

a. Show that the equation for converting base 10 logarithms to base 2 logarithms is

$$\log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x.$$

b. Show that the equation for converting base a logarithms to base b logarithms is

$$\log_b x = \frac{\ln a}{\ln b} \log_a x.$$

7.2 Exponential Change and Separable Differential Equations

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in many natural and industrial situations. The variety of models based on these functions partly accounts for their importance. We now investigate the basic proportionality assumption that leads to such *exponential change*.

Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the size of a population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium. Such quantities are said to undergo **exponential change**.

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

Differential equation: $$\frac{dy}{dt} = ky$$

Initial condition: $$y = y_0 \quad \text{when} \quad t = 0.$$

If y is positive and increasing, then k is positive, and we use Equation (1a) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1a) to say that the rate of decay is proportional to the amount still left.

Exercises 7.2

Verifying Solutions

In Exercises 1–4, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

1. $2y' + 3y = e^{-x}$
   a. $y = e^{-x}$ b. $y = e^{-x} + e^{-(3/2)x}$
   c. $y = e^{-x} + Ce^{-(3/2)x}$
2. $y' = y^2$
   a. $y = -\dfrac{1}{x}$ b. $y = -\dfrac{1}{x+3}$ c. $y = -\dfrac{1}{x+C}$
3. $y = \dfrac{1}{x}\displaystyle\int_1^x \frac{e^t}{t}\,dt,\quad x^2y' + xy = e^x$
4. $y = \dfrac{1}{\sqrt{1+x^4}}\displaystyle\int_1^x \sqrt{1+t^4}\,dt,\quad y' + \dfrac{2x^3}{1+x^4}y = 1$

Initial Value Problems

In Exercises 5–8, show that each function is a solution of the given initial value problem.

Differential equation	Initial equation	Solution candidate
5. $y' + y = \dfrac{2}{1+4e^{2x}}$	$y(-\ln 2) = \dfrac{\pi}{2}$	$y = e^{-x}\tan^{-1}(2e^x)$
6. $y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x-2)e^{-x^2}$
7. $xy' + y = -\sin x,$ $x > 0$	$y\left(\dfrac{\pi}{2}\right) = 0$	$y = \dfrac{\cos x}{x}$
8. $x^2y' = xy - y^2,$ $x > 1$	$y(e) = e$	$y = \dfrac{x}{\ln x}$

Separable Differential Equations

Solve the differential equation in Exercises 9–22.

9. $2\sqrt{xy}\dfrac{dy}{dx} = 1,\quad x, y > 0$
10. $\dfrac{dy}{dx} = x^2\sqrt{y},\quad y > 0$
11. $\dfrac{dy}{dx} = e^{x-y}$
12. $\dfrac{dy}{dx} = 3x^2 e^{-y}$
13. $\dfrac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$
14. $\sqrt{2xy}\dfrac{dy}{dx} = 1$
15. $\sqrt{x}\dfrac{dy}{dx} = e^{y+\sqrt{x}},\quad x > 0$
16. $(\sec x)\dfrac{dy}{dx} = e^{y+\sin x}$
17. $\dfrac{dy}{dx} = 2x\sqrt{1-y^2},\quad -1 < y < 1$
18. $\dfrac{dy}{dx} = \dfrac{e^{2x-y}}{e^{x+y}}$
19. $y^2\dfrac{dy}{dx} = 3x^2y^3 - 6x^2$
20. $\dfrac{dy}{dx} = xy + 3x - 2y - 6$
21. $\dfrac{1}{x}\dfrac{dy}{dx} = ye^{x^2} + 2\sqrt{y}\,e^{x^2}$
22. $\dfrac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

Applications and Examples

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

23. **Human evolution continues** The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not

In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

   a. If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0e^{kt}$. Then use this value of k to answer the following questions.
   b. In about how many years will human teeth be 90% of their present size?
   c. What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

24. **Atmospheric pressure** The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.
   a. Solve the initial value problem

      Differential equation: $dp/dh = kp$ (k a constant)
      Initial condition: $p = p_0$ when $h = 0$

      to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.
   b. What is the atmospheric pressure at $h = 50$ km?
   c. At what altitude does the pressure equal 900 millibars?

25. **First-order chemical reactions** In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,
$$\frac{dy}{dt} = -0.6y$$
when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

26. **The inversion of sugar** The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?

27. **Working underwater** The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation
$$\frac{dL}{dx} = -kL.$$
As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

28. **Voltage in a discharging capacitor** Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,
$$\frac{dV}{dt} = -\frac{1}{40}V.$$
Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

29. **Cholera bacteria** Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half-hour. How many bacteria will the colony contain at the end of 24 hours? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

30. **Growth of bacteria** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

31. **The incidence of a disease** (*Continuation of Example 4.*) Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.
    a. How long will it take to reduce the number of cases to 1000?
    b. How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

32. **Drug concentration** An antibiotic is administered intravenously into the bloodstream at a constant rate r. As the drug flows through the patient's system and acts on the infection that is present, it is removed from the bloodstream at a rate proportional to the amount in the bloodstream at that time. Since the amount of blood in the patient is constant, this means that the concentration $y = y(t)$ of the antibiotic in the bloodstream can be modeled by the differential equation

$$\frac{dy}{dt} = r - ky, \quad k > 0 \text{ and constant.}$$

    a. If $y(0) = y_0$, find the concentration $y(t)$ at any time t.
    b. Assume that $y_0 < (r/k)$ and find $\lim_{y\to\infty} y(t)$. Sketch the solution curve for the concentration.

33. **Endangered species** Biologists consider a species of animal or plant to be endangered if it is expected to become extinct within 20 years. If a certain species of wildlife is counted to have 1147 members at the present time, and the population has been steadily declining exponentially at an annual rate averaging 39% over the past 7 years, do you think the species is endangered? Explain your answer.

34. **The U.S. population** The U.S. Census Bureau keeps a running clock totaling the U.S. population. On September 20, 2012, the total was increasing at the rate of 1 person every 12 sec. The population figure for 8:11 P.M. EST on that day was 314,419,198.
    a. Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).
    b. At this rate, what will the U.S. population be at 8:11 P.M. EST on September 20, 2019?

35. **Oil depletion** Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

36. **Continuous price discounting** To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.
    a. Find $p(x)$ by solving the following initial value problem:

Differential equation:	$\frac{dp}{dx} = -\frac{1}{100}p$
Initial condition:	$p(100) = 20.09.$

    b. Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.
    c. The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.
    d. Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

37. **Plutonium-239** The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?

38. **Polonium-210** The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

39. **The mean life of a radioactive nucleus** Physicists using the radioactivity equation $y = y_0e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

40. **Californium-252** What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the Western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per minute.
    a. What is the value of k in the decay equation for this isotope?
    b. What is the isotope's mean life? (See Exercise 39.)
    c. How long will it take 95% of a sample's radioactive nuclei to disintegrate?

41. **Cooling soup** Suppose that a cup of soup cooled from 90°C to 60°C after 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.
    a. How much longer would it take the soup to cool to 35°C?
    b. Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

42. **A beam of unknown temperature** An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min it was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature.

43. **Surrounding medium of unknown temperature** A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the

water's temperature was 39°C; 10 min after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

44. **Silver cooling in air** The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be
   a. 15 min from now?
   b. 2 hours from now?
   c. When will the silver be 10°C above room temperature?

45. **The age of Crater Lake** The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

46. **The sensitivity of carbon-14 dating to measurement** To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:
   a. A bone fragment found in central Illinois in the year 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.
   b. Repeat part (a), assuming 18% instead of 17%.
   c. Repeat part (a), assuming 16% instead of 17%.

47. **Carbon-14** The oldest known frozen human mummy, discovered in the Schnalstal glacier of the Italian Alps in 1991 and called *Otzi*, was found wearing straw shoes and a leather coat with goat fur, and holding a copper ax and stone dagger. It was estimated that Otzi died 5000 years before he was discovered in the melting glacier. How much of the original carbon-14 remained in Otzi at the time of his discovery?

48. **Art forgery** A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

49. **Lascaux Cave paintings** Prehistoric cave paintings of animals were found in the Lascaux Cave in France in 1940. Scientific analysis revealed that only 15% of the original carbon-14 in the paintings remained. What is an estimate of the age of the paintings?

50. **Incan mummy** The frozen remains of a young Incan woman were discovered by archeologist Johan Reinhard on Mt. Ampato in Peru during an expedition in 1995.
   a. How much of the original carbon-14 was present if the estimated age of the "Ice Maiden" was 500 years?
   b. If a 1% error can occur in the carbon-14 measurement, what is the oldest possible age for the Ice Maiden?

7.3 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical and engineering applications. In this section we give a brief introduction to these functions, their graphs, their derivatives, their integrals, and their inverse functions.

Definitions and Identities

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

We pronounce $\sinh x$ as "cinch x," rhyming with "pinch x," and $\cosh x$ as "kosh x," rhyming with "gosh x." From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions. The defining equations and graphs of these functions are shown in Table 7.4. We will see that the hyperbolic functions bear many similarities to the trigonometric functions after which they are named.

Hyperbolic functions satisfy the identities in Table 7.5. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions, as we show here for the second one:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \sinh 2x. \end{aligned}$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$

$$= \frac{1 - 2u}{u^2} + C$$

$$= \frac{1 - 2(1 + \sqrt{x})}{(1 + \sqrt{x})^2} + C$$

$$= C - \frac{1 + 2\sqrt{x}}{(1 + \sqrt{x})^2}.$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate $\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$.

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0. \qquad \text{Theorem 8, Section 5.6}$$

Exercises 8.1

Assorted Integrations

The integrals in Exercises 1–40 are in no particular order. Evaluate each integral using any algebraic method or trigonometric identity you think is appropriate, and then use a substitution to reduce it to a standard form.

1. $\int_0^1 \frac{16x}{8x^2 + 2} dx$
2. $\int \frac{x^2}{x^2 + 1} dx$
3. $\int (\sec x - \tan x)^2 \, dx$
4. $\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$
5. $\int \frac{1 - x}{\sqrt{1 - x^2}} dx$
6. $\int \frac{dx}{x - \sqrt{x}}$
7. $\int \frac{e^{-\cot z}}{\sin^2 z} dz$
8. $\int \frac{2^{\ln z^3}}{16z} dz$
9. $\int \frac{dz}{e^z + e^{-z}}$
10. $\int_1^2 \frac{8\,dx}{x^2 - 2x + 2}$
11. $\int_{-1}^0 \frac{4\,dx}{1 + (2x + 1)^2}$
12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx$
13. $\int \frac{dt}{1 - \sec t}$
14. $\int \csc t \sin 3t \, dt$
15. $\int_0^{\pi/4} \frac{1 + \sin\theta}{\cos^2\theta} d\theta$
16. $\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$
17. $\int \frac{\ln y}{y + 4y \ln^2 y} dy$
18. $\int \frac{2^{\sqrt{y}}\,dy}{2\sqrt{y}}$
19. $\int \frac{d\theta}{\sec\theta + \tan\theta}$
20. $\int \frac{dt}{t\sqrt{3 + t^2}}$
21. $\int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt$
22. $\int \frac{x + 2\sqrt{x - 1}}{2x\sqrt{x - 1}} dx$
23. $\int_0^{\pi/2} \sqrt{1 - \cos\theta} \, d\theta$
24. $\int (\sec t + \cot t)^2 \, dt$
25. $\int \frac{dy}{\sqrt{e^{2y} - 1}}$
26. $\int \frac{6\,dy}{\sqrt{y}(1 + y)}$
27. $\int \frac{2\,dx}{x\sqrt{1 - 4\ln^2 x}}$
28. $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$
29. $\int (\csc x - \sec x)(\sin x + \cos x) \, dx$
30. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx$
31. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1} dx$
32. $\int_{-1}^1 \sqrt{1 + x^2} \sin x \, dx$
33. $\int_{-1}^0 \sqrt{\frac{1 + y}{1 - y}} dy$
34. $\int e^{z + e^z} dz$

35. $\displaystyle\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$

36. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta-1}$

39. $\displaystyle\int \frac{dx}{1+e^x}$

Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

Hint: Let $u = x^{3/2}$.

Theory and Examples

41. **Area** Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

42. **Volume** Find the volume of the solid generated by revolving the region in Exercise 41 about the x-axis.

43. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

44. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

45. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

46. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

47. The functions $y = e^{x^3}$ and $y = x^3e^{x^3}$ do not have elementary antiderivatives, but $y = (1+3x^3)e^{x^3}$ does. Evaluate

$$\int (1+3x^3)e^{x^3}\,dx.$$

48. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1+\sin^2 x}.$$

49. Use the substitution $u = x^4 + 1$ to evaluate the integral

$$\int x^7\sqrt{x^4+1}\,dx.$$

50. **Using different substitutions** Show that the integral

$$\int ((x^2-1)(x+1))^{-2/3}dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x+1)$

b. $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$

e. $u = \tan^{-1}((x-1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$

In the first case, $f(x) = \ln x$ is easy to differentiate and $g(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{n}\sin nx + \frac{3x^3}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{3\pi^2\cos n\pi}{n^2} - \frac{6\cos n\pi}{n^4} + \frac{6}{n^4}\right)$$

$$= \frac{3}{\pi}\left(\frac{\pi^2 n^2(-1)^n + 2(-1)^{n+1} + 2}{n^4}\right). \qquad \cos n\pi = (-1)^n$$

■

Integrals like those in Example 8 occur frequently in electrical engineering.

Exercises 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\,dx$
2. $\int \theta \cos \pi\theta\,d\theta$
3. $\int t^2 \cos t\,dt$
4. $\int x^2 \sin x\,dx$
5. $\int_1^2 x \ln x\,dx$
6. $\int_1^e x^3 \ln x\,dx$
7. $\int xe^x\,dx$
8. $\int xe^{3x}\,dx$
9. $\int x^2 e^{-x}\,dx$
10. $\int (x^2 - 2x + 1)e^{2x}\,dx$
11. $\int \tan^{-1} y\,dy$
12. $\int \sin^{-1} y\,dy$
13. $\int x \sec^2 x\,dx$
14. $\int 4x \sec^2 2x\,dx$
15. $\int x^3 e^x\,dx$
16. $\int p^4 e^{-p}\,dp$
17. $\int (x^2 - 5x)e^x\,dx$
18. $\int (r^2 + r + 1)e^r\,dr$
19. $\int x^5 e^x\,dx$
20. $\int t^2 e^{4t}\,dt$
21. $\int e^{\theta} \sin \theta\,d\theta$
22. $\int e^{-y} \cos y\,dy$
23. $\int e^{2x} \cos 3x\,dx$
24. $\int e^{-2x} \sin 2x\,dx$

Using Substitution

Evaluate the integrals in Exercise 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$
26. $\int_0^1 x\sqrt{1 - x}\,dx$
27. $\int_0^{\pi/3} x \tan^2 x\,dx$
28. $\int \ln (x + x^2)\,dx$
29. $\int \sin (\ln x)\,dx$
30. $\int z(\ln z)^2\,dz$

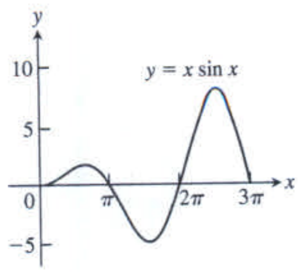
Evaluating Integrals

Evaluate the integrals in Exercises 31–52. Some integrals do not require integration by parts.

31. $\int x \sec x^2\,dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx$
33. $\int x (\ln x)^2\,dx$
34. $\int \frac{1}{x (\ln x)^2}\,dx$
35. $\int \frac{\ln x}{x^2}\,dx$
36. $\int \frac{(\ln x)^3}{x}\,dx$
37. $\int x^3 e^{x^4}\,dx$
38. $\int x^5 e^{x^3}\,dx$
39. $\int x^3\sqrt{x^2 + 1}\,dx$
40. $\int x^2 \sin x^3\,dx$
41. $\int \sin 3x \cos 2x\,dx$
42. $\int \sin 2x \cos 4x\,dx$
43. $\int \sqrt{x} \ln x\,dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
45. $\int \cos \sqrt{x}\,dx$
46. $\int \sqrt{x}\, e^{\sqrt{x}}\,dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\,d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\,dx$
49. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\,dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\,dx$
51. $\int x \tan^{-1} x\,dx$
52. $\int x^2 \tan^{-1} \frac{x}{2}\,dx$

Theory and Examples

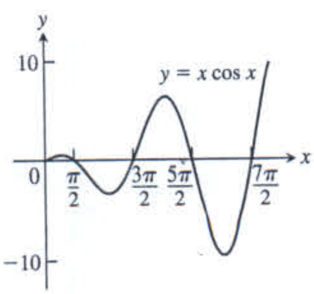
53. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for
    a. $0 \le x \le \pi$.
    b. $\pi \le x \le 2\pi$.
    c. $2\pi \le x \le 3\pi$.
    d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



54. **Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for
    a. $\pi/2 \le x \le 3\pi/2$.
    b. $3\pi/2 \le x \le 5\pi/2$.
    c. $5\pi/2 \le x \le 7\pi/2$.
    d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



55. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

56. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
    a. about the y-axis.
    b. about the line $x = 1$.

57. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
    a. the y-axis.
    b. the line $x = \pi/2$.

58. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about
    a. the y-axis.
    b. the line $x = \pi$.

    (See Exercise 53 for a graph.)

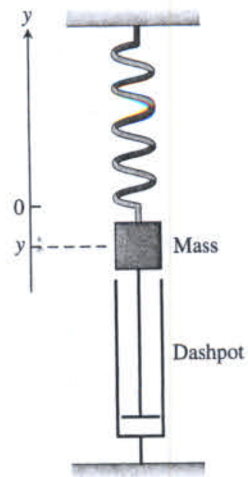
59. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.
    a. Find the area of the region.
    b. Find the volume of the solid formed by revolving this region about the x-axis.
    c. Find the volume of the solid formed by revolving this region about the line $x = -2$.
    d. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.
    a. Find the area of the region.
    b. Find the volume of the solid formed by revolving this region about the y-axis.

61. **Average value** A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



62. **Average value** In a mass-spring-dashpot system like the one in Exercise 61, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 63–67, use integration by parts to establish the reduction formula.

63. $\displaystyle\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

64. $\displaystyle\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

65. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

66. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

67. $\int x^m(\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\cdot$

$\int x^m(\ln x)^{n-1}\,dx, \quad m \neq -1$

68. Use Example 5 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \left(\frac{\pi}{2}\right)\frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots n}, & n \text{ even} \\ \frac{2\cdot4\cdot6\cdots(n-1)}{1\cdot3\cdot5\cdots n}, & n \text{ odd} \end{cases}$$

69. Show that

$$\int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx.$$

70. Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x),\ x = f(y) \\ &&& dx = f'(y)\,dy \\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with} \\ &&& u = y,\ dv = f'(y)\,dy \\ &= xf^{-1}(x) - \int f(y)\,dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x\,dx &= \int ye^y\,dy && y = \ln x,\ x = e^y \\ &&& dx = e^y\,dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C. \end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned} \int \cos^{-1} x\,dx &= x\cos^{-1} x - \int \cos y\,dy && y = \cos^{-1} x \\ &= x\cos^{-1} x - \sin y + C \\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \qquad (4)$$

to evaluate the integrals in Exercises 71–74. Express your answers in terms of x.

71. $\int \sin^{-1} x\,dx$

72. $\int \tan^{-1} x\,dx$

73. $\int \sec^{-1} x\,dx$

74. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \qquad (5)$$

Exercises 75 and 76 compare the results of using Equations (4) and (5).

75. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

76. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

b. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1+x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 77 and 78 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to x.

77. $\int \sinh^{-1} x\,dx$

78. $\int \tanh^{-1} x\,dx$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\,dx = \tan x + C.$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Exercises 8.3

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x \, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$
3. $\int \cos^3 x \sin x \, dx$
4. $\int \sin^4 2x \cos 2x \, dx$
5. $\int \sin^3 x \, dx$
6. $\int \cos^3 4x \, dx$
7. $\int \sin^5 x \, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$
9. $\int \cos^3 x \, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx$
11. $\int \sin^3 x \cos^3 x \, dx$
12. $\int \cos^3 2x \sin^5 2x \, dx$
13. $\int \cos^2 x \, dx$
14. $\int_0^{\pi/2} \sin^2 x \, dx$
15. $\int_0^{\pi/2} \sin^7 y \, dy$
16. $\int 7 \cos^7 t \, dt$
17. $\int_0^{\pi} 8 \sin^4 x \, dx$
18. $\int 8 \cos^4 2\pi x \, dx$
19. $\int 16 \sin^2 x \cos^2 x \, dx$
20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$
21. $\int 8 \cos^3 2\theta \sin 2\theta \, d\theta$
22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} \, dx$
24. $\int_0^{\pi} \sqrt{1 - \cos 2x} \, dx$
25. $\int_0^{\pi} \sqrt{1 - \sin^2 t} \, dt$
26. $\int_0^{\pi} \sqrt{1 - \cos^2 \theta} \, d\theta$
27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} \, dx$
28. $\int_0^{\pi/6} \sqrt{1 + \sin x} \, dx$

    $\left(\textit{Hint:} \text{ Multiply by } \sqrt{\frac{1 - \sin x}{1 - \sin x}}\right)$
29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1 - \sin x}} \, dx$
30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x} \, dx$
31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} \, d\theta$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} \, dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–50.

33. $\int \sec^2 x \tan x \, dx$
34. $\int \sec x \tan^2 x \, dx$
35. $\int \sec^3 x \tan x \, dx$
36. $\int \sec^3 x \tan^3 x \, dx$
37. $\int \sec^2 x \tan^2 x \, dx$
38. $\int \sec^4 x \tan^2 x \, dx$
39. $\int_{-\pi/3}^{0} 2 \sec^3 x \, dx$
40. $\int e^x \sec^3 e^x \, dx$
41. $\int \sec^4 \theta \, d\theta$
42. $\int 3 \sec^4 3x \, dx$
43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta$
44. $\int \sec^6 x \, dx$
45. $\int 4 \tan^3 x \, dx$
46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx$
47. $\int \tan^5 x \, dx$
48. $\int \cot^6 2x \, dx$
49. $\int_{\pi/6}^{\pi/3} \cot^3 x \, dx$
50. $\int 8 \cot^4 t \, dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 51–56.

51. $\int \sin 3x \cos 2x \, dx$

52. $\int \sin 2x \cos 3x \, dx$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

54. $\int_{0}^{\pi/2} \sin x \cos x \, dx$

55. $\int \cos 3x \cos 4x \, dx$

56. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Exercises 57–62 require the use of various trigonometric identities before you evaluate the integrals.

57. $\int \sin^2 \theta \cos 3\theta \, d\theta$

58. $\int \cos^2 2\theta \sin \theta \, d\theta$

59. $\int \cos^3 \theta \sin 2\theta \, d\theta$

60. $\int \sin^3 \theta \cos 2\theta \, d\theta$

61. $\int \sin \theta \cos \theta \cos 3\theta \, d\theta$

62. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 63–68.

63. $\int \frac{\sec^3 x}{\tan x} \, dx$

64. $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

65. $\int \frac{\tan^2 x}{\csc x} \, dx$

66. $\int \frac{\cot x}{\cos^2 x} \, dx$

67. $\int x \sin^2 x \, dx$

68. $\int x \cos^3 x \, dx$

Applications

69. **Arc length** Find the length of the curve

$$y = \ln(\sec x), \quad 0 \le x \le \pi/4.$$

70. **Center of gravity** Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4, x = \pi/4$.

71. **Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

72. **Area** Find the area between the x-axis and the curve $y = \sqrt{1 + \cos 4x}$, $0 \le x \le \pi$.

73. **Centroid** Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \le x \le 2\pi$.

74. **Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the x-axis.

8.4 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure 8.2.

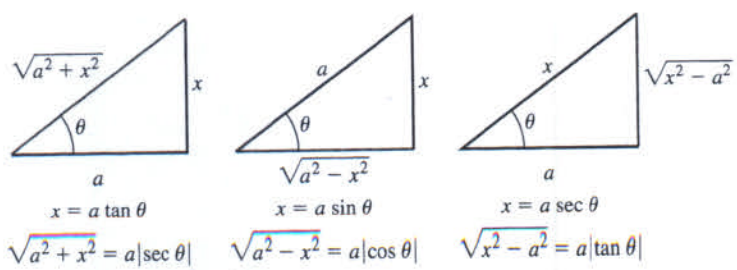


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

Exercises 8.4

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\int \frac{dx}{\sqrt{9+x^2}}$
2. $\int \frac{3\,dx}{\sqrt{1+9x^2}}$
3. $\int_{-2}^{2} \frac{dx}{4+x^2}$
4. $\int_{0}^{2} \frac{dx}{8+2x^2}$
5. $\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$
6. $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$
7. $\int \sqrt{25-t^2}\,dt$
8. $\int \sqrt{1-9t^2}\,dt$
9. $\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$
10. $\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$
11. $\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$
12. $\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$
13. $\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$
14. $\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\int \frac{x}{\sqrt{9-x^2}}\,dx$
16. $\int \frac{x^2}{4+x^2}\,dx$
17. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$
18. $\int \frac{dx}{x^2\sqrt{x^2+1}}$
19. $\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$
20. $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$
21. $\int \sqrt{\frac{x+1}{1-x}}\,dx$
22. $\int x\sqrt{x^2-4}\,dx$
23. $\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$
24. $\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$
25. $\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$
26. $\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$
27. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$
28. $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$
29. $\int \frac{8\,dx}{(4x^2+1)^2}$
30. $\int \frac{6\,dt}{(9t^2+1)^2}$
31. $\int \frac{x^3\,dx}{x^2-1}$
32. $\int \frac{x\,dx}{25+4x^2}$
33. $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$
34. $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$
36. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$
37. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$
38. $\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$
39. $\int \frac{dx}{x\sqrt{x^2-1}}$
40. $\int \frac{dx}{1+x^2}$
41. $\int \frac{x\,dx}{\sqrt{x^2-1}}$
42. $\int \frac{dx}{\sqrt{1-x^2}}$
43. $\int \frac{x\,dx}{\sqrt{1+x^4}}$
44. $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$
45. $\int \sqrt{\frac{4-x}{x}}\,dx$
   (*Hint:* Let $x = u^2$.)
46. $\int \sqrt{\frac{x}{1-x^3}}\,dx$
   (*Hint:* Let $u = x^{3/2}$.)
47. $\int \sqrt{x}\sqrt{1-x}\,dx$
48. $\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

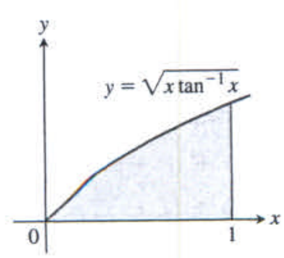
Initial Value Problems

Solve the initial value problems in Exercises 49–52 for y as a function of x.

49. $x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2, \quad y(2) = 0$
50. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$
51. $(x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$
52. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

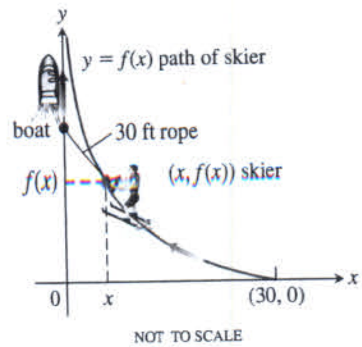
Applications and Examples

53. **Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.
54. **Area** Find the area enclosed by the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
55. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.
   a. Find the area of the region.
   b. Find the centroid of the region.
56. Consider the region bounded by the graphs of $y = \sqrt{x}\tan^{-1} x$ and $y = 0$ for $0 \leq x \leq 1$. Find the volume of the solid formed by revolving this region about the x-axis (see accompanying figure).



57. Evaluate $\int x^3 \sqrt{1 - x^2}\, dx$ using
   a. integration by parts.
   b. a u-substitution.
   c. a trigonometric substitution.

58. **Path of a water skier** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.
   a. Show that $f'(x) = \frac{-\sqrt{900 - x^2}}{x}$.
   (*Hint:* Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)
   b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



8.5 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.$$

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x^2 - 2x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x - 3}{(x + 1)(x - 3)}\, dx = \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx$$
$$= 2 \ln|x + 1| + 3 \ln|x - 3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called the **method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \tag{1}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

EXAMPLE 9 Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned} x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\ 2 &= 2A \\ A &= 1 \\ x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\ 5 &= -B \\ B &= -5 \\ x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\ 10 &= 2C \\ C &= 5. \end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

■

Exercises 8.5

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$
2. $\dfrac{5x-7}{x^2-3x+2}$
3. $\dfrac{x+4}{(x+1)^2}$
4. $\dfrac{2x+2}{x^2-2x+1}$
5. $\dfrac{z+1}{z^2(z-1)}$
6. $\dfrac{z}{z^3-z^2-6z}$
7. $\dfrac{t^2+8}{t^2-5t+6}$
8. $\dfrac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$
10. $\displaystyle\int \frac{dx}{x^2+2x}$
11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$
12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$
13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$
14. $\displaystyle\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$
15. $\displaystyle\int \frac{dt}{t^3+t^2-2t}$
16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
19. $\displaystyle\int \frac{dx}{(x^2-1)^2}$
20. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$
22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$
23. $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$
24. $\displaystyle\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$
25. $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$
26. $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$
27. $\displaystyle\int \frac{x^2-x+2}{x^3-1}\,dx$
28. $\displaystyle\int \frac{1}{x^4+x}\,dx$
29. $\displaystyle\int \frac{x^2}{x^4-1}\,dx$
30. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}\,dx$
31. $\displaystyle\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$
32. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

33. $\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

34. $\int \frac{x^4}{x^2 - 1}\,dx$

35. $\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

36. $\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

37. $\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

38. $\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 39–50.

39. $\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

40. $\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

41. $\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

42. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

43. $\int \frac{(x - 2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x - 2)^2}\,dx$

44. $\int \frac{(x + 1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x + 1)^2}\,dx$

45. $\int \frac{1}{x^{3/2} - \sqrt{x}}\,dx$

46. $\int \frac{1}{(x^{1/3} - 1)\sqrt{x}}\,dx$
(*Hint:* Let $x = u^6$.)

47. $\int \frac{\sqrt{x + 1}}{x}\,dx$
(*Hint:* Let $x + 1 = u^2$.)

48. $\int \frac{1}{x\sqrt{x + 9}}\,dx$

49. $\int \frac{1}{x(x^4 + 1)}\,dx$
$\left(\textit{Hint: } \text{Multiply by } \frac{x^3}{x^3}.\right)$

50. $\int \frac{1}{x^6(x^5 + 4)}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 51–54 for x as a function of t.

51. $(t^2 - 3t + 2)\frac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

52. $(3t^4 + 4t^2 + 1)\frac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

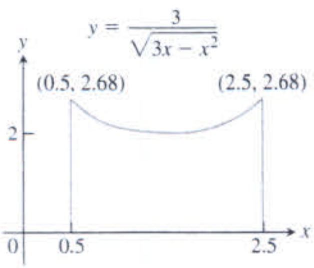
53. $(t^2 + 2t)\frac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

54. $(t + 1)\frac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

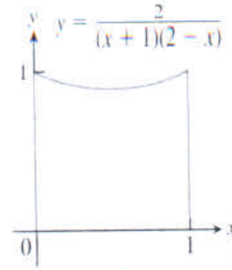
Applications and Examples

In Exercises 55 and 56, find the volume of the solid generated by revolving the shaded region about the indicated axis.

55. The x-axis

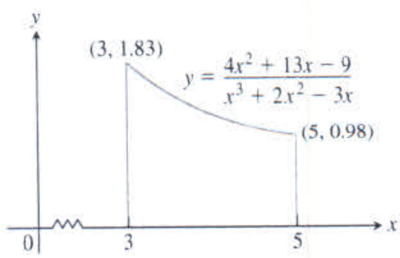


56. The y-axis



T **57.** Find, to two decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

T **58.** Find the x-coordinate of the centroid of this region to two decimal places.



T **59. Social diffusion** Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

T **60. Second-order chemical reactions** Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth.*

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ agrees to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than four, then its fourth derivative is zero, and

$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180}(0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f. In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero, and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12}(0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f. This is no surprise, for the trapezoids fit the graph perfectly.

Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice. When Δx is very small, say $\Delta x = 10^{-8}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. You should consult a text on numerical analysis for more sophisticated methods if you are having problems with round-off error using the rules discussed in this section.

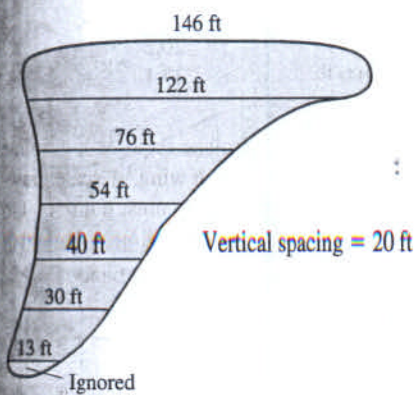


FIGURE 8.11 The dimensions of the swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a small polluted swamp (Figure 8.11). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's Rule with $\Delta x = 20$ ft and the y's equal to the distances measured across the swamp, as shown in Figure 8.11.

$$\begin{aligned} S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100 \end{aligned}$$

The volume is about $(8100)(5) = 40{,}500$ ft^3 or 1500 yd^3. ■

Exercises 8.7

Estimating Definite Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

b. Evaluate the integral directly and find $|E_T|$.

c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. Using Simpson's Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

b. Evaluate the integral directly and find $|E_S|$.

c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

1. $\int_1^2 x\,dx$

2. $\int_1^3 (2x - 1)\,dx$

3. $\int_{-1}^{1} (x^2 + 1)\,dx$

4. $\int_{-2}^{0} (x^2 - 1)\,dx$

5. $\int_{0}^{2} (t^3 + t)\,dt$

6. $\int_{-1}^{1} (t^3 + 1)\,dt$

7. $\int_{1}^{2} \frac{1}{s^2}\,ds$

8. $\int_{2}^{4} \frac{1}{(s-1)^2}\,ds$

9. $\int_{0}^{\pi} \sin t\,dt$

10. $\int_{0}^{1} \sin \pi t\,dt$

Estimating the Number of Subintervals

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

11. $\int_{1}^{2} x\,dx$

12. $\int_{1}^{3} (2x - 1)\,dx$

13. $\int_{-1}^{1} (x^2 + 1)\,dx$

14. $\int_{-2}^{0} (x^2 - 1)\,dx$

15. $\int_{0}^{2} (t^3 + t)\,dt$

16. $\int_{-1}^{1} (t^3 + 1)\,dt$

17. $\int_{1}^{2} \frac{1}{s^2}\,ds$

18. $\int_{2}^{4} \frac{1}{(s-1)^2}\,ds$

19. $\int_{0}^{3} \sqrt{x+1}\,dx$

20. $\int_{0}^{3} \frac{1}{\sqrt{x+1}}\,dx$

21. $\int_{0}^{2} \sin(x+1)\,dx$

22. $\int_{-1}^{1} \cos(x+\pi)\,dx$

Estimates with Numerical Data

23. **Volume of water in a swimming pool** A rectangular swimming pool is 30 ft wide and 50 ft long. The accompanying table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$ applied to the integral

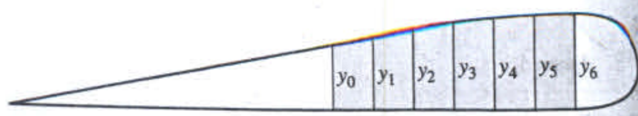
$$V = \int_{0}^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

24. **Distance traveled** The accompanying table shows time-to-speed data for a sports car accelerating from rest to 130 mph. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

Speed change	Time (sec)
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

25. **Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank by Simpson's Rule.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

26. **Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	Oil consumption rate (liters/h)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

27. **Usable values of the sine-integral function** *The sine-integral function,*

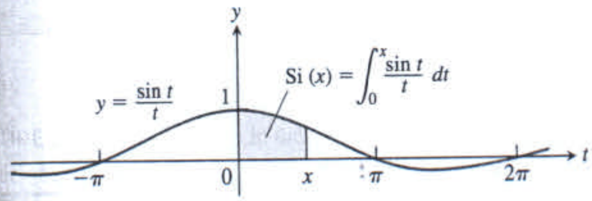
$$\mathrm{Si}(x) = \int_{0}^{x} \frac{\sin t}{t}\,dt, \qquad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\mathrm{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



a. Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's Rule with $n = 4$.

b. Estimate $\mathrm{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.

c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).

28. The error function *The error function,*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a. Use Simpson's Rule with $n = 10$ to estimate erf (1).

b. In $[0, 1]$,

$$\left|\frac{d^4}{dt^4}\left(e^{-t^2}\right)\right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

29. Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

30. Prove that the sum S in Simpson's Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 29.)

T **31. Elliptic integrals** The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

turns out to be

$$\text{Length} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t}\,dt,$$

where $e = \sqrt{a^2 - b^2}/a$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral,* is nonelementary except when $e = 0$ or 1.

a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2 \cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

Applications

T **32.** The length of one arch of the curve $y = \sin x$ is given by

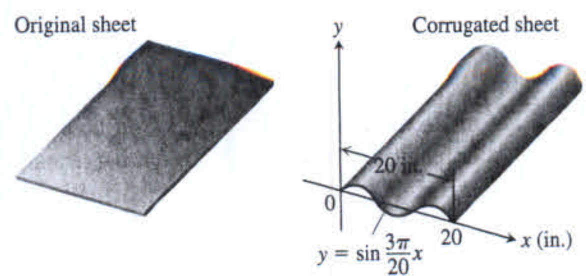
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx.$$

Estimate L by Simpson's Rule with $n = 8$.

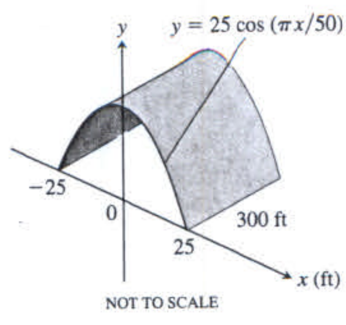
T **33.** Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20} x, \quad 0 \leq x \leq 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



T **34.** Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like one arch of the curve $y = 25 \cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$2.35 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 35 and 36 about the x-axis.

35. $y = \sin x, \quad 0 \le x \le \pi$

36. $y = x^2/4, \quad 0 \le x \le 2$

37. Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

38. Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\,dx.$$

39. **Drug assimilation** An average adult under age 60 years assimilates a 12-hr cold medicine into his or her system at a rate modeled by

$$\frac{dy}{dt} = 6 - \ln(2t^2 - 3t + 3),$$

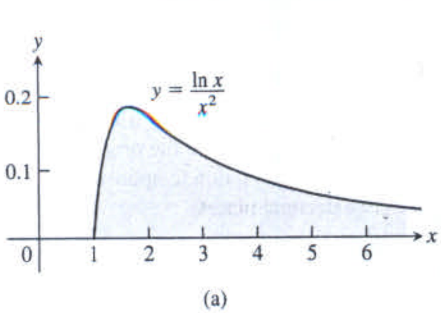
where y is measured in milligrams and t is the time in hours since the medication was taken. What amount of medicine is absorbed into a person's system over a 12-hr period?

40. **Effects of an antihistamine** The concentration of an antihistamine in the bloodstream of a healthy adult is modeled by

$$C = 12.5 - 4\ln(t^2 - 3t + 4),$$

where C is measured in grams per liter and t is the time in hours since the medication was taken. What is the average level of concentration in the bloodstream over a 6-hr period?

8.8 Improper Integrals



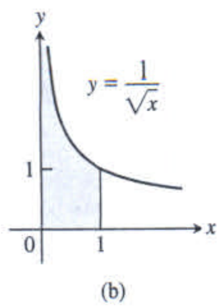


FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

Up to now, we have required definite integrals to have two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.12a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.12b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see in Section 8.9 that improper integrals play an important role in probability. They are also useful when investigating the convergence of certain infinite series in Chapter 10.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.13a). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.13b).

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$