Exercises 8.8

Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^1 \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^\infty \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^\infty \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\displaystyle\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^0 \theta e^\theta\,d\theta$

22. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^1 x\ln x\,dx$

26. $\displaystyle\int_0^1 (-\ln x)\,dx$

27. $\displaystyle\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\displaystyle\int_0^1 \frac{\ln x}{x^2}\,dx$

38. $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

39. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\displaystyle\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$

42. $\displaystyle\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

43. $\displaystyle\int_0^2 \frac{dx}{1-x^2}$

44. $\displaystyle\int_0^2 \frac{dx}{1-x}$

45. $\displaystyle\int_{-1}^1 \ln|x|\,dx$

46. $\displaystyle\int_{-1}^1 -x\ln|x|\,dx$

47. $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$

48. $\displaystyle\int_4^\infty \frac{dx}{\sqrt{x}-1}$

49. $\displaystyle\int_2^\infty \frac{dv}{\sqrt{v-1}}$

50. $\displaystyle\int_0^\infty \frac{d\theta}{1+e^\theta}$

51. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$

52. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

53. $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\displaystyle\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$

56. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$

57. $\displaystyle\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

58. $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$

59. $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$

60. $\displaystyle\int_{e^e}^\infty \ln(\ln x)\,dx$

61. $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\displaystyle\int_1^\infty \frac{1}{e^x-2^x}\,dx$

63. $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

64. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

 a. $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ b. $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

66. **$\int_{-\infty}^\infty f(x)\,dx$ may not equal $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$** Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^b \frac{2x\,dx}{x^2+1} = 0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.

70. Find the volume of the solid generated by revolving the region about the x-axis.

71. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

72. The region in Exercise 71 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

73. Evaluate the integrals.

a. $\displaystyle\int_0^1 \frac{dt}{\sqrt{t}(1+t)}$ **b.** $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}(1+t)}$

74. Evaluate $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2-9}}$.

75. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

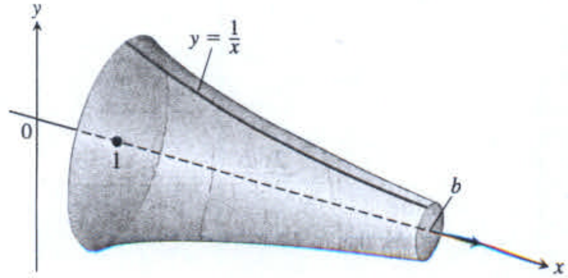
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

76. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi\int_1^b \frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

a. Calculate it.

b. This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

77. Sine-integral function The integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the *sine-integral function*, has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the sine-integral function everywhere increasing or decreasing? Do you think $\mathrm{Si}(x) = 0$ for $x > 0$? Check your answers by graphing the function $\mathrm{Si}(x)$ for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

78. Error function The function

$$\mathrm{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.4, Exercise 41.

79. Normal probability distribution The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean. (See Section 8.9.)

From the theory of probability, it is known that

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^n f(x)\,dx$$

for $n = 1, 2$, and 3.

c. Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_{b}^{\infty} e^{-x/2}\,dx \to 0 \quad \text{as} \quad b \to \infty.)$$

80. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

b. $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

COMPUTER EXPLORATIONS

In Exercises 81–84, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

81. $\displaystyle\int_{0}^{e} x^p \ln x\,dx$

82. $\displaystyle\int_{e}^{\infty} x^p \ln x\,dx$

83. $\displaystyle\int_{0}^{\infty} x^p \ln x\,dx$

84. $\displaystyle\int_{-\infty}^{\infty} x^p \ln |x|\,dx$

Use a CAS to evaluate the integrals.

85. $\displaystyle\int_{0}^{2/\pi} \sin \frac{1}{x}\,dx$

86. $\displaystyle\int_{0}^{2/\pi} x \sin \frac{1}{x}\,dx$

8.9 Probability

The outcome of some events, such as a heavy rock falling from a great height, can be modeled so that we can predict with high accuracy what will happen. On the other hand, many events have more than one possible outcome and which one of them will occur is uncertain. If we toss a coin, a head or a tail will result with each outcome being equally likely, but we do not know in advance which one it will be. If we randomly select and then weigh a person from a large population, there are many possible weights the person might have, and it is not certain whether the weight will be between 180 and 190 lb. We are told it is highly likely, but not known for sure, that an earthquake of magnitude 6.0 or greater on the Richter scale will occur near a major population area in California within the next one hundred years. Events having more than one possible outcome are *probabilistic* in nature, and when modeling them we assign a *probability* to the likelihood that a particular outcome may occur. In this section we show how calculus plays a central role in making predictions with probabilistic models.

Random Variables

We begin our discussion with some familiar examples of uncertain events for which the collection of all possible outcomes is finite.

EXAMPLE 1

(a) If we toss a coin once, there are two possible outcomes $\{H, T\}$, where H represents the coin landing head face up and T a tail landing face up. If we toss a coin three times, there are eight possible outcomes, taking into account the order in which a head or tail occurs. The set of outcomes is $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

(b) If we roll a six-sided die once, the set of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$ representing the six faces of the die.

(c) If we select at random two cards from a 52-card deck, there are 52 possible outcomes for the first card drawn and then 51 possibilities for the second card. Since the order of the cards does not matter, there are $(52 \cdot 51)/2 = 1{,}326$ possible outcomes altogether. ■

It is customary to refer to the set of all possible outcomes as the *sample space* for an event. With an uncertain event we are usually interested in which outcomes, if any, are more likely to occur than others, and to how large an extent. In tossing a coin three times,

Exercises 11.1

Finding Cartesian from Parametric Equations

Exercises 1–18 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$
2. $x = -\sqrt{t}, \quad y = t, \quad t \geq 0$
3. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$
4. $x = 3 - 3t, \quad y = 2t, \quad 0 \leq t \leq 1$
5. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \leq t \leq \pi$
6. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \leq t \leq \pi$
7. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \leq t \leq 2\pi$
8. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \leq t \leq 2\pi$
9. $x = \sin t, \quad y = \cos 2t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
10. $x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \leq t \leq \pi$
11. $x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$
12. $x = \dfrac{t}{t-1}, \quad y = \dfrac{t-2}{t+1}, \quad -1 < t < 1$
13. $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \leq t \leq 0$
14. $x = \sqrt{t+1}, \quad y = \sqrt{t}, \quad t \geq 0$
15. $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
16. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$
17. $x = -\cosh t, \quad y = \sinh t, \quad -\infty < t < \infty$
18. $x = 2\sinh t, \quad y = 2\cosh t, \quad -\infty < t < \infty$

Finding Parametric Equations

19. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$
    - a. once clockwise.
    - b. once counterclockwise.
    - c. twice clockwise.
    - d. twice counterclockwise.

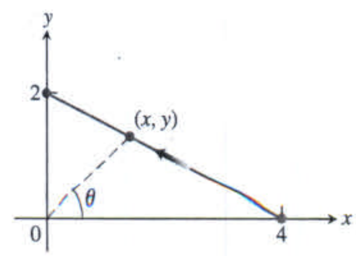
    (There are many ways to do these, so your answers may not be the same as the ones in the back of the book.)

20. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$
    - a. once clockwise.
    - b. once counterclockwise.
    - c. twice clockwise.
    - d. twice counterclockwise.

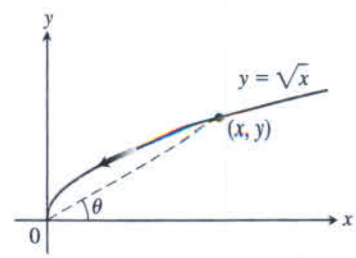
    (As in Exercise 19, there are many correct answers.)

In Exercises 21–26, find a parametrization for the curve.

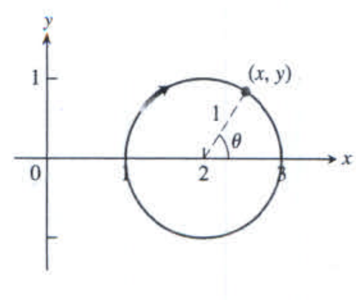
21. the line segment with endpoints $(-1, -3)$ and $(4, 1)$
22. the line segment with endpoints $(-1, 3)$ and $(3, -2)$
23. the lower half of the parabola $x - 1 = y^2$
24. the left half of the parabola $y = x^2 + 2x$
25. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$
26. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$
27. Find parametric equations and a parameter interval for the motion of a particle starting at the point $(2, 0)$ and tracing the top half of the circle $x^2 + y^2 = 4$ four times.
28. Find parametric equations and a parameter interval for the motion of a particle that moves along the graph of $y = x^2$ in the following way: Beginning at $(0, 0)$ it moves to $(3, 9)$, and then travels back and forth from $(3, 9)$ to $(-3, 9)$ infinitely many times.
29. Find parametric equations for the semicircle
$$x^2 + y^2 = a^2, \quad y > 0,$$
using as parameter the slope $t = dy/dx$ of the tangent to the curve at (x, y).
30. Find parametric equations for the circle
$$x^2 + y^2 = a^2,$$
using as parameter the arc length s measured counterclockwise from the point $(a, 0)$ to the point (x, y).
31. Find a parametrization for the line segment joining points $(0, 2)$ and $(4, 0)$ using the angle θ in the accompanying figure as the parameter.



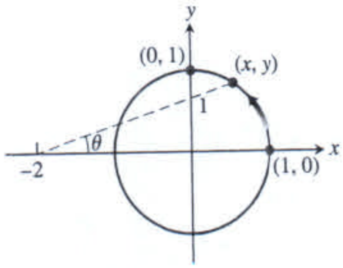
32. Find a parametrization for the curve $y = \sqrt{x}$ with terminal point $(0, 0)$ using the angle θ in the accompanying figure as the parameter.



33. Find a parametrization for the circle $(x - 2)^2 + y^2 = 1$ starting at $(1, 0)$ and moving clockwise once around the circle, using the central angle θ in the accompanying figure as the parameter.



34. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at $(1, 0)$ and moving counterclockwise to the terminal point $(0, 1)$, using the angle θ in the accompanying figure as the parameter.

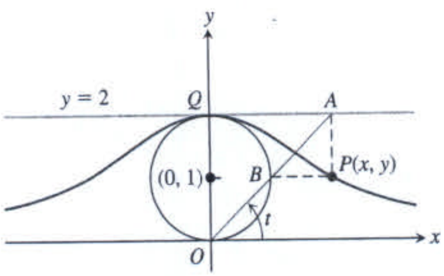


35. The witch of Maria Agnesi The bell-shaped witch of Maria Agnesi can be constructed in the following way. Start with a circle of radius 1, centered at the point $(0, 1)$, as shown in the accompanying figure. Choose a point A on the line $y = 2$ and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$. Find parametric equations and a parameter interval for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment OA makes with the positive x-axis. The following equalities (which you may assume) will help.

a. $x = AQ$

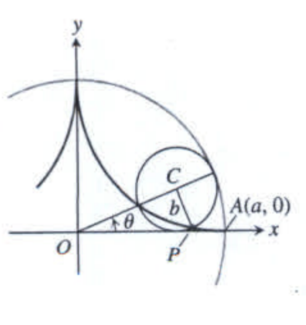
b. $y = 2 - AB \sin t$

c. $AB \cdot OA = (AQ)^2$

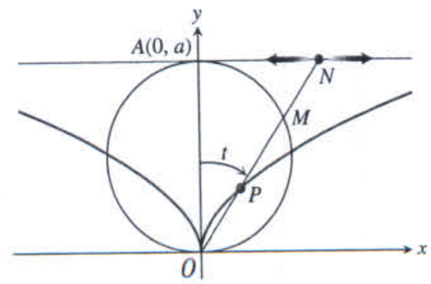


36. Hypocycloid When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a *hypocycloid*. Let the fixed circle be $x^2 + y^2 = a^2$, let the radius of the rolling circle be b, and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the hypocycloid, using as the parameter the angle θ from the positive x-axis to the line joining the circles' centers. In particular, if $b = a/4$, as in the accompanying figure, show that the hypocycloid is the astroid

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta.$$



37. As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Find parametric equations for the coordinates of P as functions of the angle t that the line ON makes with the positive y-axis.



38. Trochoids A wheel of radius a rolls along a horizontal straight line without slipping. Find parametric equations for the curve traced out by a point P on a spoke of the wheel b units from its center. As parameter, use the angle θ through which the wheel turns. The curve is called a *trochoid*, which is a cycloid when $b = a$.

Distance Using Parametric Equations

39. Find the point on the parabola $x = t, y = t^2, -\infty < t < \infty$, closest to the point $(2, 1/2)$. (*Hint:* Minimize the square of the distance as a function of t.)

40. Find the point on the ellipse $x = 2 \cos t, y = \sin t, 0 \le t \le 2\pi$ closest to the point $(3/4, 0)$. (*Hint:* Minimize the square of the distance as a function of t.)

T GRAPHER EXPLORATIONS

If you have a parametric equation grapher, graph the equations over the given intervals in Exercises 41–48.

41. Ellipse $x = 4 \cos t, \quad y = 2 \sin t,$ over

a. $0 \le t \le 2\pi$

b. $0 \le t \le \pi$

c. $-\pi/2 \le t \le \pi/2.$

42. Hyperbola branch $x = \sec t$ (enter as $1/\cos(t)$), $y = \tan t$ (enter as $\sin(t)/\cos(t)$), over

a. $-1.5 \le t \le 1.5$

b. $-0.5 \le t \le 0.5$

c. $-0.1 \le t \le 0.1.$

43. Parabola $x = 2t + 3, \quad y = t^2 - 1, \quad -2 \le t \le 2$

44. Cycloid $x = t - \sin t, \quad y = 1 - \cos t,$ over

a. $0 \le t \le 2\pi$

b. $0 \le t \le 4\pi$

c. $\pi \le t \le 3\pi.$

45. Deltoid

$$x = 2 \cos t + \cos 2t, \quad y = 2 \sin t - \sin 2t; \quad 0 \le t \le 2\pi$$

What happens if you replace 2 with -2 in the equations for x and y? Graph the new equations and find out.

46. A nice curve

$$x = 3 \cos t + \cos 3t, \quad y = 3 \sin t - \sin 3t; \quad 0 \le t \le 2\pi$$

What happens if you replace 3 with -3 in the equations for x and y? Graph the new equations and find out.

47. a. **Epicycloid**

$x = 9\cos t - \cos 9t, \quad y = 9\sin t - \sin 9t; \quad 0 \le t \le 2\pi$

b. **Hypocycloid**

$x = 8\cos t + 2\cos 4t, \quad y = 8\sin t - 2\sin 4t; \quad 0 \le t \le 2\pi$

c. **Hypotrochoid**

$x = \cos t + 5\cos 3t, \quad y = 6\cos t - 5\sin 3t; \quad 0 \le t \le 2\pi$

48. a. $x = 6\cos t + 5\cos 3t, \quad y = 6\sin t - 5\sin 3t;$
$0 \le t \le 2\pi$

b. $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 2t - 5\sin 6t;$
$0 \le t \le \pi$

c. $x = 6\cos t + 5\cos 3t, \quad y = 6\sin 2t - 5\sin 3t;$
$0 \le t \le 2\pi$

d. $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 4t - 5\sin 6t;$
$0 \le t \le \pi$

11.2 Calculus with Parametric Curves

In this section we apply calculus to parametric curves. Specifically, we find slopes, lengths, and areas associated with parametrized curves.

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

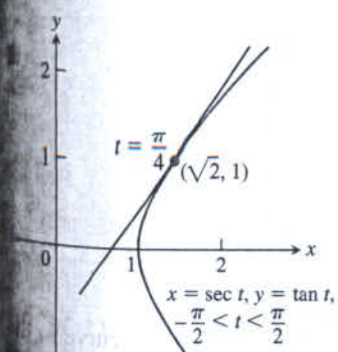
If parametric equations define y as a twice-differentiable function of x, we can apply Equation (1) to the function $dy/dx = y'$ to calculate d^2y/dx^2 as a function of t:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}. \qquad \text{Eq. (1) with } y' \text{ in place of } y$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{2}$$



GURE 11.13 The curve in Example 1 the right-hand branch of the hyperbola $- y^2 = 1$.

EXAMPLE 1 Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Figure 11.13).

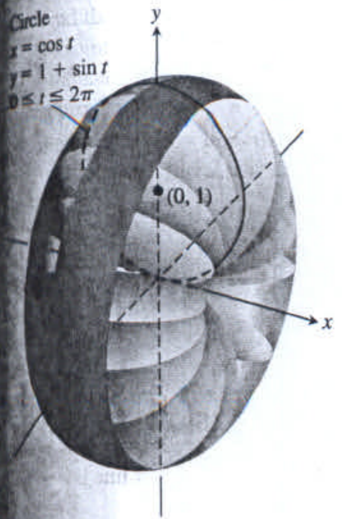


FIGURE 11.18 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

EXAMPLE 9 The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis (Figure 11.18).

Solution We evaluate the formula

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} 2\pi(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt \\ &= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt \\ &= 2\pi \Big[t - \cos t \Big]_0^{2\pi} = 4\pi^2. \end{aligned}$$

Eq. (5) for revolution about the x-axis; $y = 1 + \sin t \ge 0$

■

Exercises 11.2

Tangents to Parametrized Curves

In Exercises 1–14, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

1. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$
2. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$
3. $x = 4\sin t, \quad y = 2\cos t, \quad t = \pi/4$
4. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$
5. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$
6. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$
7. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$
8. $x = -\sqrt{t+1}, \quad y = \sqrt{3t}, \quad t = 3$
9. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$
10. $x = 1/t, \quad y = -2 + \ln t, \quad t = 1$
11. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$
12. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$
13. $x = \dfrac{1}{t+1}, \quad y = \dfrac{t}{t-1}, \quad t = 2$
14. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Implicitly Defined Parametrizations

Assuming that the equations in Exercises 15–20 define x and y implicitly as differentiable functions $x = f(t)$, $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t.

15. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$
16. $x = \sqrt{5 - \sqrt{t}}, \quad y(t-1) = \sqrt{t}, \quad t = 4$
17. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$
18. $x \sin t + 2x = t, \quad t\sin t - 2t = y, \quad t = \pi$
19. $x = t^3 + t, \quad y + 2t^3 = 2x + t^2, \quad t = 1$
20. $t = \ln(x - t), \quad y = te^t, \quad t = 0$

Area

21. Find the area under one arch of the cycloid
$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$
22. Find the area enclosed by the y-axis and the curve
$$x = t - t^2, \quad y = 1 + e^{-t}.$$
23. Find the area enclosed by the ellipse
$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$
24. Find the area under $y = x^3$ over $[0, 1]$ using the following parametrizations.
 a. $x = t^2, \quad y = t^6$
 b. $x = t^3, \quad y = t^9$

Lengths of Curves

Find the lengths of the curves in Exercises 25–30.

25. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$
26. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$
27. $x = t^2/2, \quad y = (2t + 1)^{3/2}/3, \quad 0 \le t \le 4$
28. $x = (2t + 3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$
29. $x = 8\cos t + 8t\sin t$
 $y = 8\sin t - 8t\cos t,$
 $0 \le t \le \pi/2$
30. $x = \ln(\sec t + \tan t) - \sin t$
 $y = \cos t, \quad 0 \le t \le \pi/3$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 31–34 about the indicated axes.

31. $x = \cos t, \quad y = 2 + \sin t, \quad 0 \le t \le 2\pi;$ x-axis

32. $x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}$; y-axis

33. $x = t + \sqrt{2}, \quad y = (t^2/2) + \sqrt{2}t, \ -\sqrt{2} \le t \le \sqrt{2}$; y-axis

34. $x = \ln(\sec t + \tan t) - \sin t,\ y = \cos t,\ 0 \le t \le \pi/3$; x-axis

35. A cone frustum The line segment joining the points (0, 1) and (2, 2) is revolved about the x-axis to generate a frustum of a cone. Find the surface area of the frustum using the parametrization $x = 2t, y = t + 1, 0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi(r_1 + r_2)$(slant height).

36. A cone The line segment joining the origin to the point (h, r) is revolved about the x-axis to generate a cone of height h and base radius r. Find the cone's surface area with the parametric equations $x = ht,\ y = rt,\ 0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi r$(slant height).

Centroids

37. Find the coordinates of the centroid of the curve

$$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad 0 \le t \le \pi/2.$$

38. Find the coordinates of the centroid of the curve

$$x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \le t \le \pi.$$

39. Find the coordinates of the centroid of the curve

$$x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi.$$

T **40.** Most centroid calculations for curves are done with a calculator or computer that has an integral evaluation program. As a case in point, find, to the nearest hundredth, the coordinates of the centroid of the curve

$$x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}.$$

Theory and Examples

41. Length is independent of parametrization To illustrate the fact that the numbers we get for length do not depend on the way we parametrize our curves (except for the mild restrictions preventing doubling back mentioned earlier), calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations:

a. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2.$

b. $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2.$

42. a. Show that the Cartesian formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the length of the curve $x = g(y), c \le y \le d$ (Section 6.3, Equation 4), is a special case of the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Use this result to find the length of each curve.

b. $x = y^{3/2}, \quad 0 \le y \le 4/3$

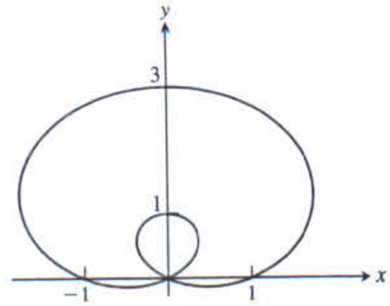
c. $x = \frac{3}{2}y^{2/3}, \quad 0 \le y \le 1$

43. The curve with parametric equations

$$x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta$$

is called a *limaçon* and is shown in the accompanying figure. Find the points (x, y) and the slopes of the tangent lines at these points for

a. $\theta = 0$. **b.** $\theta = \pi/2$. **c.** $\theta = 4\pi/3$.

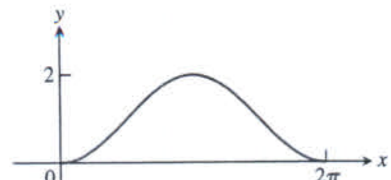


44. The curve with parametric equations

$$x = t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$$

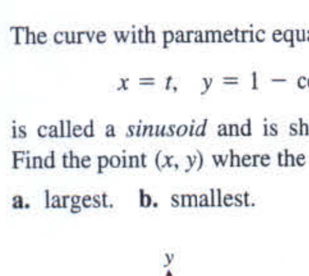
is called a *sinusoid* and is shown in the accompanying figure. Find the point (x, y) where the slope of the tangent line is

a. largest. **b.** smallest.

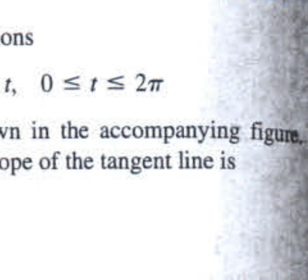


T The curves in Exercises 45 and 46 are called *Bowditch curves* or *Lissajous figures*. In each case, find the point in the interior of the first quadrant where the tangent to the curve is horizontal, and find the equations of the two tangents at the origin.

45.



46.



47. Cycloid

a. Find the length of one arch of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

b. Find the area of the surface generated by revolving one arch of the cycloid in part (a) about the x-axis for $a = 1$.

48. Volume Find the volume swept out by revolving the region bounded by the x-axis and one arch of the cycloid

$$x = t - \sin t, \quad y = 1 - \cos t$$

about the x-axis.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to perform the following steps for the given curve over the closed interval.

a. Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 11.15.)

b. Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

c. Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

49. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1$

50. $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3, \quad 0 \le t \le 6$

51. $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi$

52. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi$

11.3 Polar Coordinates

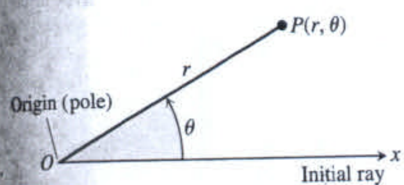


FIGURE 11.19 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

In this section we study polar coordinates and their relation to Cartesian coordinates. Yo will see that polar coordinates are very useful for calculating many multiple integrals stud ied in Chapter 15. They are also useful in describing the paths of planets and satellites.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ra** from O (Figure 11.19). Usually the positive x-axis is chosen as the initial ray. Then eac point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r give the directed distance from O to P and θ gives the directed angle from the initial ray to ra OP. So we label the point P as

$$P(r, \theta)$$

Directed distance from O to P — Directed angle from initial ray to OP

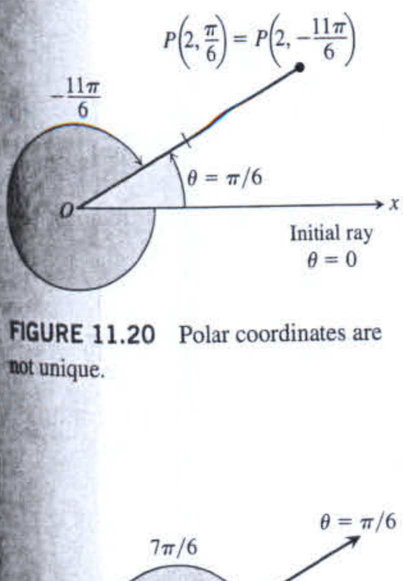


FIGURE 11.20 Polar coordinates are not unique.

As in trigonometry, θ is positive when measured counterclockwise and negative whe measured clockwise. The angle associated with a given point is not unique. While a poin in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of pola coordinates. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ ha polar coordinates $r = 2,\ \theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$ (Figur 11.20). In some situations we allow r to be negative. That is why we use directed distanc in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians coun terclockwise from the initial ray and going forward 2 units (Figure 11.21). It can also b reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backwar* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.

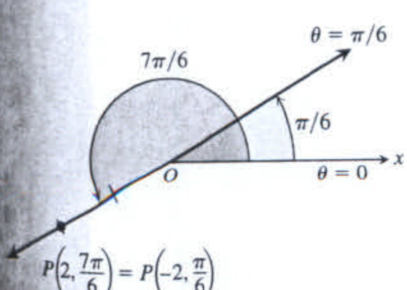


FIGURE 11.21 Polar coordinates can have negative r-values.

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the or gin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6$ (Figure 11.22). We then find the angles for the other coordinate pairs of P in which $r =$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6},\quad \frac{\pi}{6} \pm 2\pi,\quad \frac{\pi}{6} \pm 4\pi,\quad \frac{\pi}{6} \pm 6\pi, \ldots.$$

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:	$r^2 = 4r\cos\theta$	
	$x^2 + y^2 = 4x$	Substitution
	$x^2 - 4x + y^2 = 0$	
	$x^2 - 4x + 4 + y^2 = 4$	Completing the square
	$(x-2)^2 + y^2 = 4$	Factoring

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:	$r(2\cos\theta - \sin\theta) = 4$	
	$2r\cos\theta - r\sin\theta = 4$	Multiplying by r
	$2x - y = 4$	Substitution
	$y = 2x - 4$	Solve for y.

The graph: Line, slope $m = 2$, y-intercept $b = -4$

Exercises 11.3

Polar Coordinates

1. Which polar coordinate pairs label the same point?

a. $(3, 0)$ **b.** $(-3, 0)$ **c.** $(2, 2\pi/3)$
d. $(2, 7\pi/3)$ **e.** $(-3, \pi)$ **f.** $(2, \pi/3)$
g. $(-3, 2\pi)$ **h.** $(-2, -\pi/3)$

2. Which polar coordinate pairs label the same point?

a. $(-2, \pi/3)$ **b.** $(2, -\pi/3)$ **c.** (r, θ)
d. $(r, \theta + \pi)$ **e.** $(-r, \theta)$ **f.** $(2, -2\pi/3)$
g. $(-r, \theta + \pi)$ **h.** $(-2, 2\pi/3)$

3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(2, \pi/2)$ **b.** $(2, 0)$
c. $(-2, \pi/2)$ **d.** $(-2, 0)$

4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(3, \pi/4)$ **b.** $(-3, \pi/4)$
c. $(3, -\pi/4)$ **d.** $(-3, -\pi/4)$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.

6. Find the Cartesian coordinates of the following points (given in polar coordinates).

a. $(\sqrt{2}, \pi/4)$ **b.** $(1, 0)$
c. $(0, \pi/2)$ **d.** $(-\sqrt{2}, \pi/4)$
e. $(-3, 5\pi/6)$ **f.** $(5, \tan^{-1}(4/3))$
g. $(-1, 7\pi)$ **h.** $(2\sqrt{3}, 2\pi/3)$

Cartesian to Polar Coordinates

7. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(1, 1)$ **b.** $(-3, 0)$
c. $(\sqrt{3}, -1)$ **d.** $(-3, 4)$

8. Find the polar coordinates, $-\pi \le \theta < \pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(-2, -2)$ **b.** $(0, 3)$
c. $(-\sqrt{3}, 1)$ **d.** $(5, -12)$

9. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \le 0$, of the following points given in Cartesian coordinates.

a. $(3, 3)$ **b.** $(-1, 0)$
c. $(-1, \sqrt{3})$ **d.** $(4, -3)$

10. Find the polar coordinates, $-\pi \le \theta < 2\pi$ and $r \le 0$, of the following points given in Cartesian coordinates.

a. $(-2, 0)$ **b.** $(1, 0)$
c. $(0, -3)$ **d.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Graphing Sets of Polar Coordinate Points

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 11–26.

11. $r = 2$ **12.** $0 \le r \le 2$
13. $r \ge 1$ **14.** $1 \le r \le 2$

15. $0 \le \theta \le \pi/6, \quad r \ge 0$

16. $\theta = 2\pi/3, \quad r \le -2$

17. $\theta = \pi/3, \quad -1 \le r \le 3$

18. $\theta = 11\pi/4, \quad r \ge -1$

19. $\theta = \pi/2, \quad r \ge 0$

20. $\theta = \pi/2, \quad r \le 0$

21. $0 \le \theta \le \pi, \quad r = 1$

22. $0 \le \theta \le \pi, \quad r = -1$

23. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$

24. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$

25. $-\pi/2 \le \theta \le \pi/2, \quad 1 \le r \le 2$

26. $0 \le \theta \le \pi/2, \quad 1 \le |r| \le 2$

Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations. Then describe or identify the graph.

27. $r\cos\theta = 2$

28. $r\sin\theta = -1$

29. $r\sin\theta = 0$

30. $r\cos\theta = 0$

31. $r = 4\csc\theta$

32. $r = -3\sec\theta$

33. $r\cos\theta + r\sin\theta = 1$

34. $r\sin\theta = r\cos\theta$

35. $r^2 = 1$

36. $r^2 = 4r\sin\theta$

37. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

38. $r^2\sin 2\theta = 2$

39. $r = \cot\theta\csc\theta$

40. $r = 4\tan\theta\sec\theta$

41. $r = \csc\theta\, e^{r\cos\theta}$

42. $r\sin\theta = \ln r + \ln\cos\theta$

43. $r^2 + 2r^2\cos\theta\sin\theta = 1$

44. $\cos^2\theta = \sin^2\theta$

45. $r^2 = -4r\cos\theta$

46. $r^2 = -6r\sin\theta$

47. $r = 8\sin\theta$

48. $r = 3\cos\theta$

49. $r = 2\cos\theta + 2\sin\theta$

50. $r = 2\cos\theta - \sin\theta$

51. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$

52. $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 53–66 with equival polar equations.

53. $x = 7$

54. $y = 1$

55. $x = y$

56. $x - y = 3$

57. $x^2 + y^2 = 4$

58. $x^2 - y^2 = 1$

59. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

60. $xy = 2$

61. $y^2 = 4x$

62. $x^2 + xy + y^2 = 1$

63. $x^2 + (y - 2)^2 = 4$

64. $(x - 5)^2 + y^2 = 25$

65. $(x - 3)^2 + (y + 1)^2 = 4$

66. $(x + 2)^2 + (y - 5)^2 = 16$

67. Find all polar coordinates of the origin.

68. Vertical and horizontal lines

a. Show that every vertical line in the xy-plane has a polar ec tion of the form $r = a\sec\theta$.

b. Find the analogous polar equation for horizontal lines in xy-plane.

11.4 Graphing Polar Coordinate Equations

It is often helpful to graph an equation expressed in polar coordinates in the Cartesian plane. This section describes some techniques for graphing these equations using symı tries and tangents to the graph.

Symmetry

Figure 11.27 illustrates the standard polar coordinate tests for symmetry. The follow summary says how the symmetric points are related.

Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.27a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.27b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.27c).

Converting a Graph from the $r\theta$- to xy-Plane

One way to graph a polar equation $r = f(\theta)$ in the xy-plane is to make a table of (r, θ)-values, plot the corresponding points there, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing is to

1. first graph the function $r = f(\theta)$ in the *Cartesian* $r\theta$-plane,
2. then use that Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph in the xy-plane.

This method is sometimes better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where r is positive, negative, and nonexistent, as well as where r is increasing and decreasing. Here's an example.

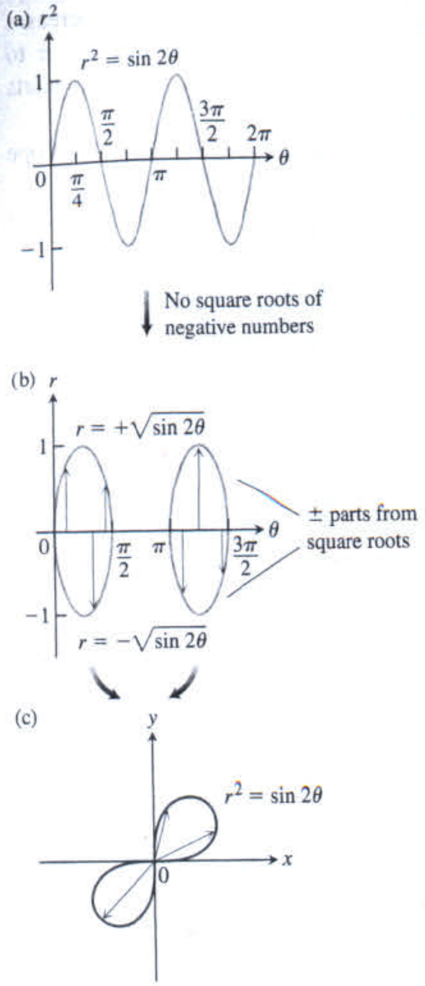


FIGURE 11.30 To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a) and then ignore the values of θ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

EXAMPLE 3 Graph the *lemniscate* curve $r^2 = \sin 2\theta$ in the Cartesian xy-plane.

Solution Here we begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane. See Figure 11.30a. We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane (Figure 11.30b), and then draw the polar graph (Figure 11.30c). The graph in Figure 11.30b "covers" the final polar graph in Figure 11.30c twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way. ■

USING TECHNOLOGY Graphing Polar Curves Parametrically

For complicated polar curves we may need to use a graphing calculator or computer to graph the curve. If the device does not plot polar graphs directly, we can convert $r = f(\theta)$ into parametric form using the equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then we use the device to draw a parametrized curve in the Cartesian xy-plane. It may be necessary to use the parameter t rather than θ for the graphing device.

Exercises 11.4

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves in the xy-plane.

1. $r = 1 + \cos\theta$
2. $r = 2 - 2\cos\theta$
3. $r = 1 - \sin\theta$
4. $r = 1 + \sin\theta$
5. $r = 2 + \sin\theta$
6. $r = 1 + 2\sin\theta$
7. $r = \sin(\theta/2)$
8. $r = \cos(\theta/2)$
9. $r^2 = \cos\theta$
10. $r^2 = \sin\theta$
11. $r^2 = -\sin\theta$
12. $r^2 = -\cos\theta$

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

13. $r^2 = 4\cos 2\theta$
14. $r^2 = 4\sin 2\theta$
15. $r^2 = -\sin 2\theta$
16. $r^2 = -\cos 2\theta$

Slopes of Polar Curves in the xy-Plane

Find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. **Cardioid** $r = -1 + \cos\theta$; $\theta = \pm\pi/2$
18. **Cardioid** $r = -1 + \sin\theta$; $\theta = 0, \pi$
19. **Four-leaved rose** $r = \sin 2\theta$; $\theta = \pm\pi/4, \pm 3\pi/4$
20. **Four-leaved rose** $r = \cos 2\theta$; $\theta = 0, \pm\pi/2, \pi$

Graphing Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

21. Limaçons with an inner loop

a. $r = \frac{1}{2} + \cos\theta$ **b.** $r = \frac{1}{2} + \sin\theta$

22. Cardioids

a. $r = 1 - \cos\theta$ **b.** $r = -1 + \sin\theta$

23. Dimpled limaçons

a. $r = \frac{3}{2} + \cos\theta$ **b.** $r = \frac{3}{2} - \sin\theta$

24. Oval limaçons

a. $r = 2 + \cos\theta$ **b.** $r = -2 + \sin\theta$

Graphing Polar Regions and Curves in the *xy*-Plane

25. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

26. Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

27. $0 \le r \le 2 - 2\cos\theta$ **28.** $0 \le r^2 \le \cos\theta$

T **29.** Which of the following has the same graph as $r = 1 - \cos\theta$?

a. $r = -1 - \cos\theta$ **b.** $r = 1 + \cos\theta$

Confirm your answer with algebra.

T **30.** Which of the following has the same graph as $r = \cos 2\theta$?

a. $r = -\sin(2\theta + \pi/2)$ **b.** $r = -\cos(\theta/2)$

Confirm your answer with algebra.

T **31. A rose within a rose** Graph the equation $r = 1 - 2\sin 3\theta$.

T **32. The nephroid of Freeth** Graph the nephroid of Freeth:

$$r = 1 + 2\sin\frac{\theta}{2}.$$

T **33. Roses** Graph the roses $r = \cos m\theta$ for $m = 1/3, 2, 3$, and 7.

T **34. Spirals** Polar coordinates are just the thing for defining spirals. Graph the following spirals.

a. $r = \theta$

b. $r = -\theta$

c. *A logarithmic spiral:* $r = e^{\theta/10}$

d. *A hyperbolic spiral:* $r = 8/\theta$

e. *An equilateral hyperbola:* $r = \pm 10/\sqrt{\theta}$

(Use different colors for the two branches.)

T **35.** Graph the equation $r = \sin\left(\frac{8}{7}\theta\right)$ for $0 \le \theta \le 14\pi$.

T **36.** Graph the equation

$$r = \sin^2(2.3\theta) + \cos^4(2.3\theta)$$

for $0 \le \theta \le 10\pi$.

11.5 Areas and Lengths in Polar Coordinates

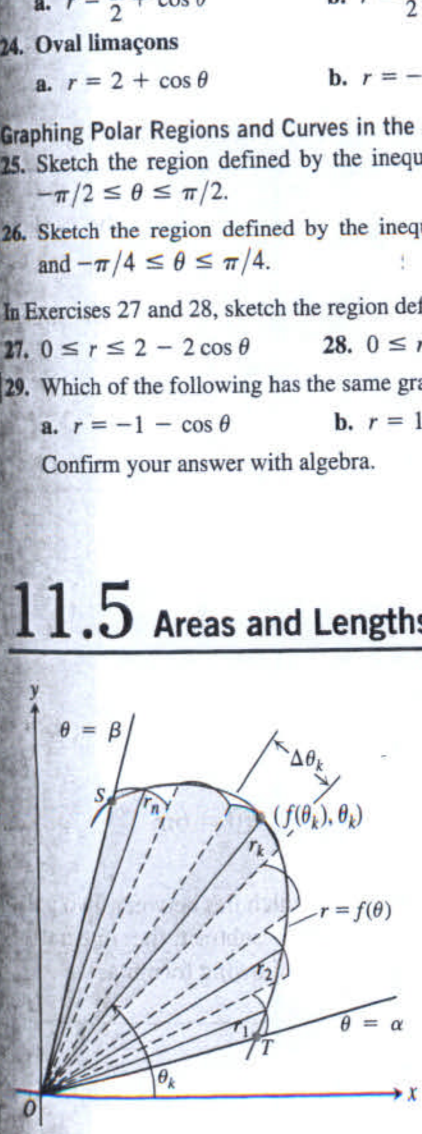


FIGURE 11.31 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

This section shows how to calculate areas of plane regions and lengths of curves in polar coordinates. The defining ideas are the same as before, but the formulas are different in polar versus Cartesian coordinates.

Area in the Plane

The region *OTS* in Figure 11.31 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2}r_k^{\,2}\,\Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n}\frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. We are then led to the following formula defining the region's area: