Math 21B (Winter 2006)
Kouba
Exam 1

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM SOMEONE ELSE'S EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 7 pages, including the cover page.

6. You will be graded on proper use of integral and derivative notation.

7. You will be graded on proper use of limit notation.

8. You have until 11:50 a.m. to finish the exam.

1.) (8 pts. each) Integrate each of the following. DO NOT SIMPLIFY answers.

a.) $\int \sqrt{x}(1+x)\ dx = \int \left(x^{1/2} + x^{3/2}\right) dx$

$= \dfrac{x^{3/2}}{3/2} + \dfrac{x^{5/2}}{5/2} + c$

b.) $\int \dfrac{x^2+3}{x+2}\ dx = \int \left(x - 2 + \dfrac{7}{x+2}\right) dx$

$$\begin{array}{r|l} & x - 2 \\ \hline x+2 & x^2 + 3 \\ & -(x^2 + 2x) \\ \hline & -2x + 3 \\ & -(-2x - 4) \\ \hline & 7 \end{array}$$

$= \dfrac{x^2}{2} - 2x + 7 \ln|x+2| + c$

c.) $\int \dfrac{e^x}{e^x+5}\ dx$ $\quad$ (Let $u = e^x + 5 \rightarrow du = e^x dx$)

$= \int \dfrac{1}{u}\, du = \ln|u| + c = \ln|e^x + 5| + c$

d.) $\int \dfrac{x}{(3x^2+1)^4}\ dx$ $\quad$ (Let $u = 3x^2 + 1 \rightarrow du = 6x\, dx \rightarrow$ $\frac{1}{6} du = dx$)

$= \frac{1}{6} \int \frac{1}{u^4} du = \frac{1}{6} \int u^{-4} du = \frac{1}{6} \cdot \frac{u^{-3}}{-3} + c$

$= \frac{-1}{18} (3x^2 + 1)^{-3} + c$

e.) $\int \sin^2 x\ \cos x\ dx$ $\quad$ (Let $u = \sin x \rightarrow du = \cos x\, dx$)

$= \int u^2 du = \frac{1}{3} u^3 + c = \frac{1}{3} \sin^3 x + c$

2.) Consider the function $f(x) = xe^{x^4}$.

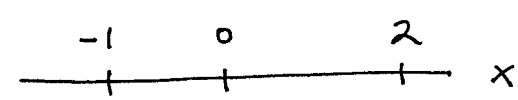
a.) (6 pts.) Verify that f is an odd function. Show $f(-x) = -f(x)$:

$$f(-x) = (-x)\cdot e^{(-x)^4}$$
$$= -x e^{x^4}$$
$$= -f(x)$$

b.) (4 pts.) Evaluate $\int_{-3}^{3} xe^{x^4}\,dx$. Since $f(x) = xe^{x^4}$ is odd,

$$\int_{-3}^{3} xe^{x^4}\,dx = 0$$

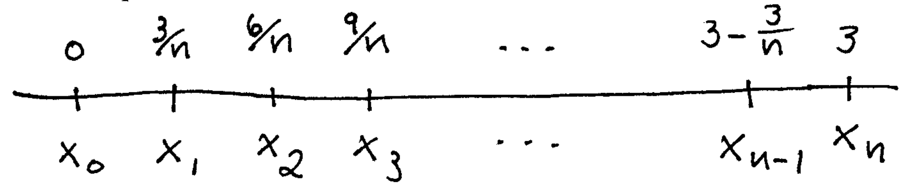
3.) (6 pts.) If $\int_{-1}^{2} f(x)\,dx = 3$ and $\int_{0}^{2} f(x)\,dx = -4$, what is $\int_{0}^{-1} f(x)\,dx$?



By properties of a definite integral:

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx \rightarrow$$
$$3 = \int_{-1}^{0} f(x)\,dx + -4 \rightarrow$$
$$\int_{-1}^{0} f(x)\,dx = 7 \rightarrow$$
$$\int_{0}^{-1} f(x)\,dx = -7.$$

4.) (12 pts.) Use the limit definition of the definite integral (for convenience, you may choose equal subdivisions and right-hand endpoints) to evaluate $\int_0^3 (x^2 + 2x)\, dx$.



$f(x) = x^2 + 2x$

Divide $[0,3]$ into n equal parts each of length $3/n$. Then right-hand endpoints are

$$x_i = 0 + \frac{3}{n} i = \frac{3}{n} i \text{ with } \Delta x_i = \frac{3}{n} \text{ for } i = 1, 2, 3, \ldots, n.$$

$$\int_0^3 (x^2 + 2x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[x_i^2 + 2x_i\right]\cdot \Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{9}{n^2} i^2 + \frac{6}{n} i\right]\cdot \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27}{n^3} i^2 + \frac{18}{n^2} i\right)$$

$$= \lim_{n\to\infty} \left\{ \frac{27}{n^3}\cdot\left(\sum_{i=1}^{n} i^2\right) + \frac{18}{n^2}\cdot\left(\sum_{i=1}^{n} i\right)\right\}$$

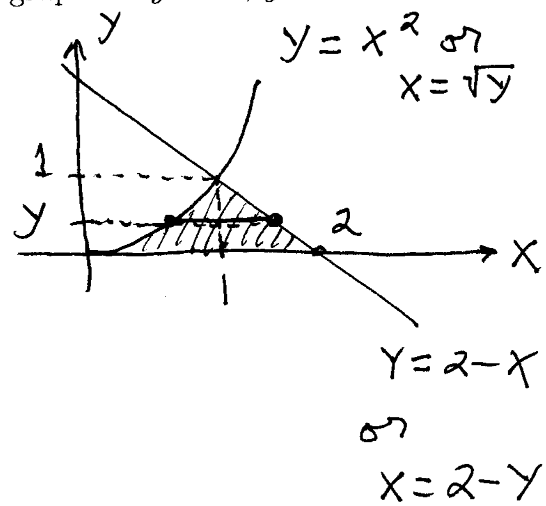
$$= \lim_{n\to\infty} \left\{ \frac{27}{n^3}\cdot \frac{n(n+1)(2n+1)}{6} + \frac{18}{n^2}\cdot\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{9}{2}\cdot\frac{\cancel{n}}{\cancel{n}}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} + 9\cdot\frac{\cancel{n}}{\cancel{n}}\cdot\frac{n+1}{n}\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{9}{2}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 9\cdot\left(1+\frac{1}{n}\right)\right\}$$

$$= \frac{9}{2}\cdot(1)(2) + 9(1) = 18$$

5.) (8 pts.) Compute the area of the region bounded by the graphs of $y = x^2$, $y = 2 - x$, and $y = 0$.



$$\text{Area} = \int_0^1 \left[(2-y) - \sqrt{y}\,\right] dy$$

$$= \left(2y - \frac{y^2}{2} - \frac{2}{3} y^{3/2}\right)\Big|_0^1$$

$$= 2 - \frac{1}{2} - \frac{2}{3}$$

$$= \frac{12}{6} - \frac{3}{6} - \frac{4}{6}$$

$$= \frac{5}{6}$$

6.) (8 pts.) The temperature T of a room at time t minutes is $T(t) = \sqrt{16 + t}\ \ {}^oF$. Find the *average* temperature of the room from $t = 0$ to $t = 20$ minutes.

$$\text{AVE} = \frac{1}{20 - 0} \int_0^{20} \sqrt{16 + t}\, dt$$

$$= \frac{1}{20} \cdot \frac{2}{3} (16 + t)^{3/2} \Big|_0^{20}$$

$$= \frac{1}{30} \left(36^{3/2} - 16^{3/2}\right)$$

$$= \frac{1}{30} (216 - 64)$$

$$= \frac{152}{30} \approx 5.07\ {}^\circ F$$

7.) (5 pts. each) Use FTC1 to differentiate each function.

a.) $F(x) = \int_x^4 \cos\sqrt{t}\,dt$ $= -\int_4^x \cos\sqrt{t}\,dt \xrightarrow{D}$

$F'(x) = -\cos\sqrt{x}$

b.) $F(x) = \int_{x^2}^{3x} e^{t^2}\,dt.$ $= \int_{x^2}^{0} e^{t^2}\,dt + \int_0^{3x} e^{t^2}\,dt$

$= -\int_0^{x^2} e^{t^2}\,dt + \int_0^{3x} e^{t^2}\,dt \xrightarrow{D}$

$F'(x) = -e^{(x^2)^2}\cdot(2x) + e^{(3x)^2}\cdot(3)$

$= -2xe^{x^4} + 3e^{9x^2}$

8.) (6 pts.) Write the following limit as a definite integral, then evaluate the integral. HINT : First identify x_i.

$\lim_{n\to\infty} \sum_{i=1}^{n} \left\{3 + \frac{2i}{n}\right\}\frac{4}{n}$ Let $x_i = 3 + \frac{2i}{n}$ for $i = 1, 2, 3, \ldots, n$;

$3 \quad 3+\frac{2}{n} \quad 3+\frac{4}{n} \quad 3+\frac{6}{n} \quad \ldots \quad 3+\frac{2n}{n} = 5$

$x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_n$

$\Delta x_i = \frac{2}{n}$ for $i = 1, 2, 3, \ldots, n$; then

$\lim_{n\to\infty} \sum_{i=1}^{n} \left(3 + \frac{2i}{n}\right)\cdot\frac{4}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(3+\frac{2i}{n}\right)\cdot\frac{2}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i \cdot \Delta x_i = \int_3^5 2x\,dx = x^2\Big|_3^5 = 5^2 - 3^2 = 16$

The following EXTRA CREDIT problem is OPTIONAL. It is worth 10 points.

1.) Use the limit definition of the definite integral, $\lim_{mesh \to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$, to evaluate $\int_a^b \frac{1}{x^2}\, dx$. Use an arbitrary partition $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ for the interval $[a, b]$ and sampling numbers $c_i = \sqrt{x_{i-1}x_i}$ for $i = 1, 2, 3, \ldots, n$.

$$\int_a^b \frac{1}{x^2}\, dx = \lim_{mesh \to 0} \sum_{i=1}^{n} f(\sqrt{x_{i-1}x_i}) \cdot (x_i - x_{i-1})$$

$$= \lim_{mesh \to 0} \sum_{i=1}^{n} \frac{1}{x_{i-1}\, x_i} (x_i - x_{i-1})$$

$$= \lim_{mesh \to 0} \sum_{i=1}^{n} \left(\frac{x_i}{x_{i-1}x_i} - \frac{x_{i-1}}{x_{i-1}x_i} \right)$$

$$= \lim_{mesh \to 0} \sum_{i=1}^{n} \left(\frac{1}{x_{i-1}} - \frac{1}{x_i} \right)$$

$$= \lim_{mesh \to 0} \left\{ \left(\frac{1}{x_0} - \cancel{\frac{1}{x_1}}\right) + \left(\cancel{\frac{1}{x_1}} - \cancel{\frac{1}{x_2}}\right) + \left(\cancel{\frac{1}{x_2}} - \cancel{\frac{1}{x_3}}\right) + \cdots + \left(\cancel{\frac{1}{x_{n-2}}} - \cancel{\frac{1}{x_{n-1}}}\right) + \left(\cancel{\frac{1}{x_{n-1}}} - \frac{1}{x_n}\right) \right\}$$

$$= \lim_{mesh \to 0} \left\{ \frac{1}{a} - \frac{1}{b} \right\}$$

$$= \frac{1}{a} - \frac{1}{b}$$