Math 21C
Kouba
Absolute Convergence Test

Absolute Convergence Test : Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$, which has both positive and negative terms. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges ($< \infty$), then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Proof : Consider that for $n = 1, 2, 3, 4, \cdots$

$$a_n + |a_n| = \begin{cases} 2 \cdot |a_n| \ , & \text{if } a_n > 0 \\ 0 \ , & \text{if } a_n < 0 \ . \end{cases}$$

Thus,

$$0 \le a_n + |a_n| \le 2 \cdot |a_n|$$

for $n = 1, 2, 3, 4, \cdots$. But the series $\displaystyle\sum_{n=1}^{\infty} 2 \cdot |a_n| = 2 \cdot \sum_{n=1}^{\infty} |a_n|$ converges (Since scalar multiples of convergent series are convergent.), so that $\displaystyle\sum_{n=1}^{\infty} (\ a_n + |a_n|\ )$ converges by the Comparison Test. We also have that $\displaystyle\sum_{n=1}^{\infty} -|a_n|$ converges (Since scalar multiples of convergent series are convergent.). It follows that

$$\sum_{n=1}^{\infty} (\ (a_n + |a_n|) \ + \ (-|a_n|)\ ) = \sum_{n=1}^{\infty} a_n$$

converges since the sum of convergent series is convergent. This completes the proof.