Math 21C
Kouba
Complex Numbers

Def: $i = \sqrt{-1}$.

Then $i^2 = (\sqrt{-1})^2 = -1$,

$i^3 = i^2 \cdot i = -i$,

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$,

$i^5 = i^4 \cdot i = (1)i = i$,

$i^6 = i^4 \cdot i^2 = (1)(-1) = -1$,

$i^7 = i^4 \cdot i^3 = (1)(-i) = -i$,

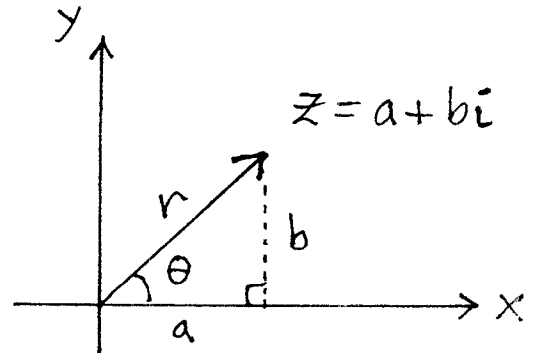$i^8 = i^4 \cdot i^4 = (1)(1) = 1$, ...

Ex: $i^{83} = (i^4)^{20} \cdot i^3 = (1)^{20}(-i) = -i$

Ex: Solve $x^2 + 1 = 0$. Then

$x^2 = -1 \implies x = i$ or $x = -i$.

Def: A complex number is any number of the form $z = a + bi$, where a and b are real numbers.

Ex: $3 - i$, $2 + 7i$, $-4i$, $\frac{7}{2}$ are all complex numbers.

Note: The complex number $z = a + bi$ can be represented in the xy-plane as a vector from the origin with length

r and direction θ. From polar coordinates we know that

$$\begin{cases} a = r \cos\theta \\ b = r \sin\theta \end{cases} \quad \text{and} \quad r = \sqrt{a^2 + b^2}. \quad \text{Then}$$

and $\boxed{Z = a + bi}$ (rectangular form)

$$Z = r\cos\theta + r\sin\theta \cdot i \implies$$

$$\boxed{Z = r(\cos\theta + i\sin\theta)} \quad (\text{polar form}).$$

Def: The __magnitude__ (length) of $Z = a + bi$ is $|Z| = \sqrt{a^2 + b^2}$. The __argument__ (direction) of $Z = a + bi$ is θ, where $\tan\theta = b/a$.

Note : 1.) If $Z = \cos\theta + i\sin\theta$, then

$$|Z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$$

2.) If $Z = a + bi$, then $|Z| = \sqrt{a^2 + b^2}$ and

$$w = \frac{Z}{|Z|} = \frac{1}{\sqrt{a^2+b^2}}(a+bi) = \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i$$

has the same direction as Z and $|w| = 1$; if we let $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$, then $w = \cos\theta + i\sin\theta$.
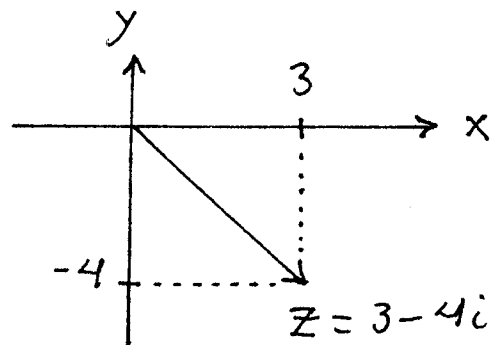
Ex: Simplify each and write answers in the form $Z = a + bi$; plot Z and

compute $|z|$.

1.) $(2+i) + (1-5i)$:

$(2+i) + (1-5i) = 3-4i$;

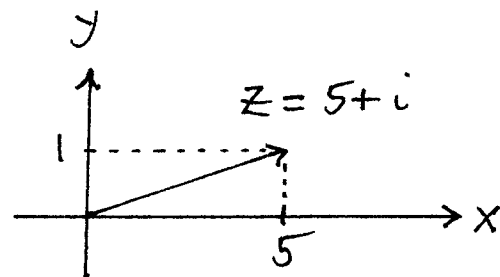$|3-4i| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

$z = 3-4i$

2.) $(3-2i)(1+i)$:

$(3-2i)(1+i) = 3 + 3i - 2i - 2i^2$

$= 3 + i - 2(-1) = 5 + i$;

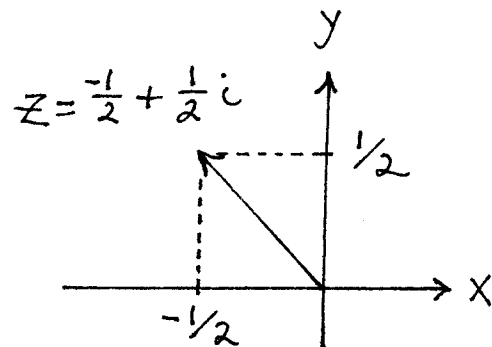$|5+i| = \sqrt{5^2 + 1^2} = \sqrt{26}$

$z = 5 + i$

3.) $\dfrac{i}{1-i}$:

$\dfrac{i}{1-i} = \dfrac{i}{1-i} \cdot \dfrac{1+i}{1+i} = \dfrac{i + i^2}{1^2 - i^2}$

$= \dfrac{i-1}{1-(-1)} = \dfrac{i-1}{2} = \dfrac{-1}{2} + \dfrac{1}{2} i$;

$\left| \dfrac{-1}{2} + \dfrac{1}{2} i \right| = \sqrt{\left(\dfrac{-1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2} = \sqrt{\dfrac{1}{2}} = \dfrac{1}{\sqrt{2}}$
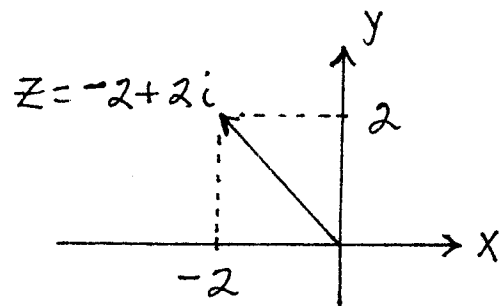

$z = \dfrac{-1}{2} + \dfrac{1}{2} i$

4.) $(1+i)^3$:

$(1+i)^3 = 1^3 + 3i + 3i^2 + i^3$

$= 1 + 3i - 3 - i = -2 + 2i$;

$|-2 + 2i| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$z = -2 + 2i$

5.) $(1+i)^{20}$?! *   (need new tools)

Theorem: Let $z = r_1 (\cos \theta_1 + i \sin \theta_1)$ and
$w = r_2 (\cos \theta_2 + i \sin \theta_2)$. Then the magnitude
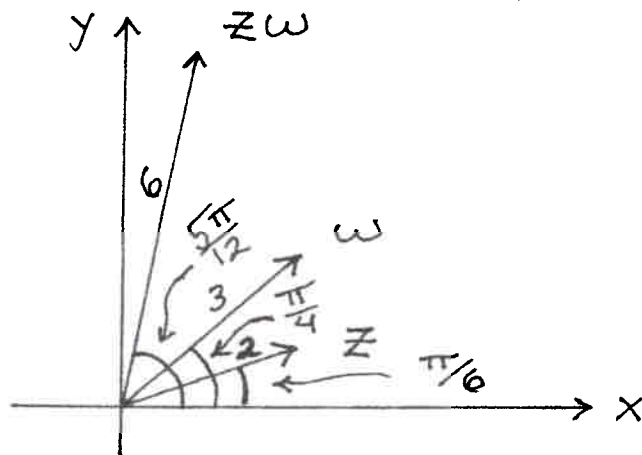of zw is $|zw| = r_1 r_2$ and the argument

is argument $(zw) = \theta_1 + \theta_2$.
(multiply lengths, add angles)

Proof: $zw = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2$
$\quad + i\sin\theta_1 \cos\theta_2 + i^2 \sin\theta_1 \sin\theta_2]$

$= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$
$\quad + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)]$

$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

Fact: If $z = r(\cos\theta + i\sin\theta)$, then
$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$.
(De moivre's law)

Ex: If $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and
$w = 3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, then
$zw = 6(\cos(\frac{\pi}{6} + \frac{\pi}{4}) + i\sin(\frac{\pi}{6} + \frac{\pi}{4}))$
$\quad = 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$.

Ex: Let $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. Compute and plot $z, z^2, z^3, z^4,$ and z^5. Write answers in rectangular form.

$$Z = 2\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right) = 2\left(\tfrac{\sqrt{3}}{2} + i\cdot\tfrac{1}{2}\right) = \sqrt{3} + i \;;$$

$$Z^2 = 2^2\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)^2 = 4\left(\cos\tfrac{2\pi}{6} + i\sin\tfrac{2\pi}{6}\right)$$
$$= 4\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) = 4\left(\tfrac{1}{2} + i\cdot\tfrac{\sqrt{3}}{2}\right) = 2 + 2\sqrt{3}\,i \;;$$

$$Z^3 = 2^3\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)^3 = 8\left(\cos\tfrac{3\pi}{6} + i\sin\tfrac{3\pi}{6}\right)$$
$$= 8\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right) = 8(0 + i) = 8i \;;$$

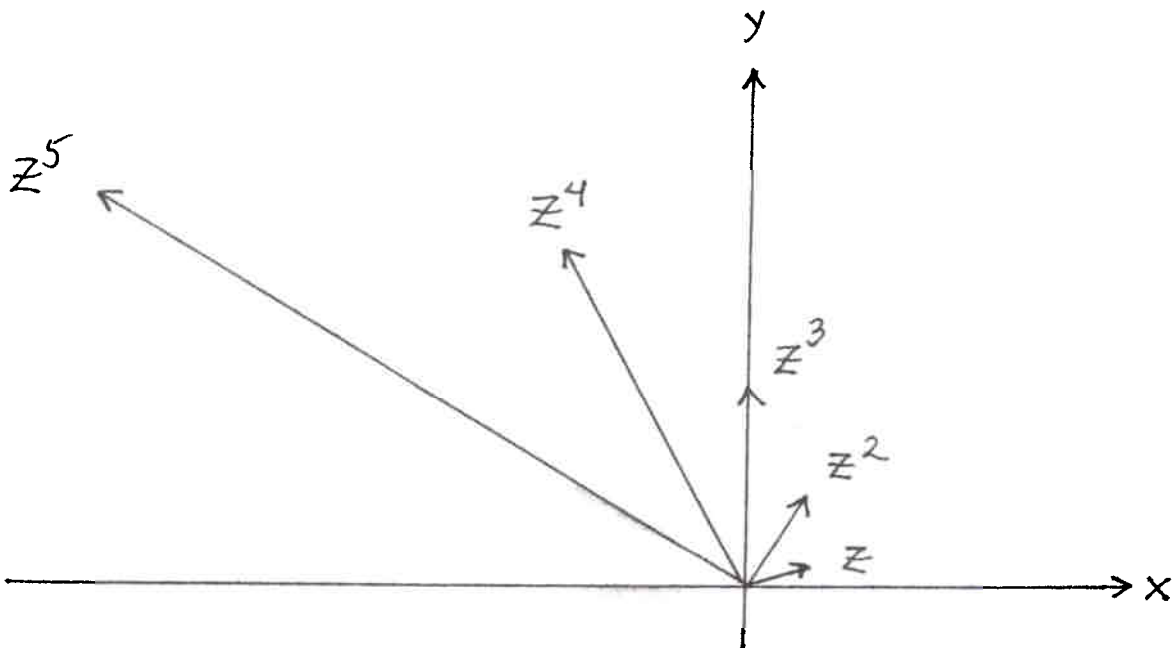$$Z^4 = 2^4\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)^4 = 16\left(\cos\tfrac{4\pi}{6} + i\sin\tfrac{4\pi}{6}\right)$$
$$= 16\left(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}\right) = 16\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) = -8 + 8\sqrt{3}\,i \;;$$

$$Z^5 = 2^5\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)^5 = 32\left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right)$$
$$= 32\left(-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right) = -16\sqrt{3} + 16i \;; \dots$$

$$Z^n = 2^n\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)^n = 2^n\left(\cos\tfrac{n\pi}{6} + i\sin\tfrac{n\pi}{6}\right) \text{ has}$$
magnitude 2^n and argument $\tfrac{n\pi}{6}$.

__EX:__ Simplify $(1+i)^{20}$ and write answer in rectangular form.

Let $Z = 1 + i$, then $|Z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\Rightarrow$

$$Z = 1 + i = \sqrt{2} \cdot \frac{1+i}{\sqrt{2}}$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$= \sqrt{2}\left(\cos\theta + i\sin\theta\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad ;$$

then

$$(1+i)^{20} = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{20}$$

$$= \left(\sqrt{2}\right)^{20}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{20}$$

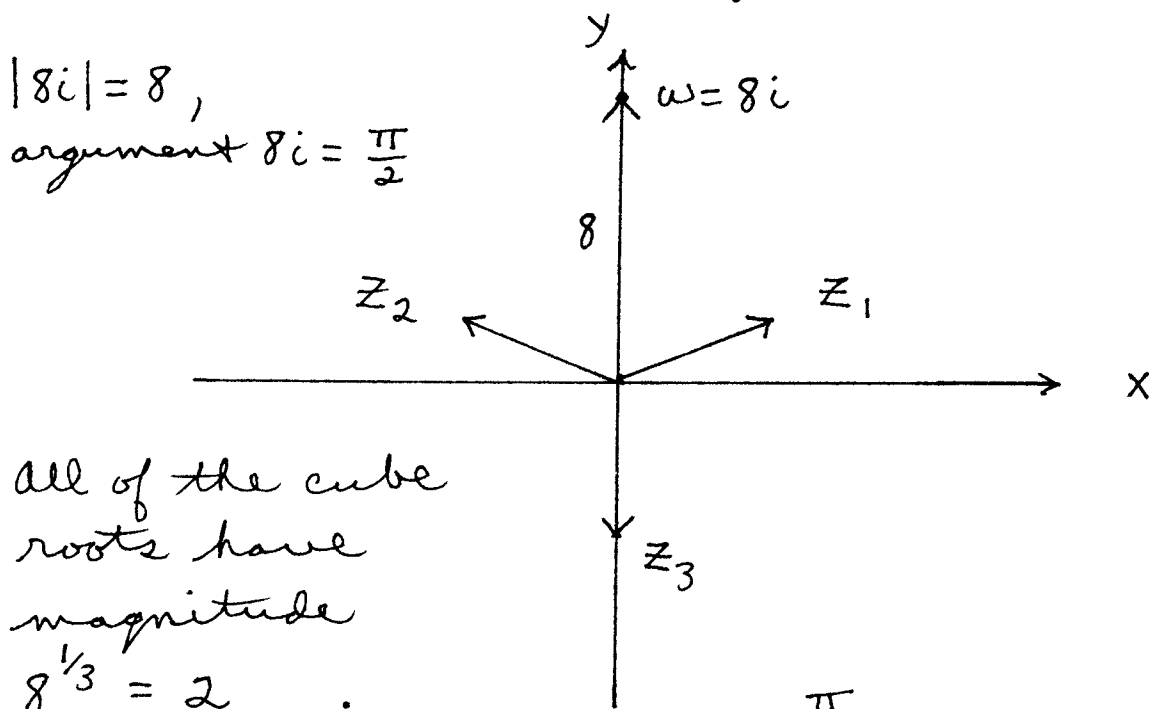$$= 2^{10}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right)$$

$$= 1024\left(\cos 5\pi + i\sin 5\pi\right)$$

$$= 1024\left(-1 + i\cdot(0)\right)$$

$$= -1024 \quad .$$

Fact: Let $w = a + bi$ be a fixed complex number. Then the equation $z^n = w$ has n equally-spaced solutions of equal length, where n is a positive integer.

Ex: Solve $z^3 = 8i$ for z, i.e., find all three cube roots of $8i$:

$|8i| = 8$,

argument $8i = \frac{\pi}{2}$

all of the cube roots have magnitude $8^{1/3} = 2$.

The argument of $z_1 = \frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$. The other cube roots are equally spaced, so there is an angle of $\frac{2\pi}{3}$ between consecutive roots:

$$z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = \sqrt{3} + i \; ;$$

$$Z_2 = 2\left(\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right)$$

$$= 2\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = -\sqrt{3} + i \;;$$

$$Z_3 = 2\left(\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)\right)$$

$$= 2\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right)$$

$$= 2\left(0 + i \cdot (-1)\right) = -2i \quad.$$