Math 21C
Kouba
Complex Numbers and Taylor Series

Recall : 1.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x-values

2.) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all x-values

3.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for all x-values

Question : Is there a mathematical connection between e^x and the trig functions $\cos x$ and $\sin x$? Yes ...

Consider the complex number $z = i\theta$ where θ is a real number. Then

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta + (i^2)\frac{\theta^2}{2!} + \frac{(i^3)\theta^3}{3!} + \frac{(i^4)\theta^4}{4!} + \frac{(i^5)\theta^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\cdot\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\cdot\frac{\theta^5}{5!} - \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta \quad , \text{ i.e.,}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ for all real numbers θ.

Ex: Simplify each expression.

1.) $4e^{i\frac{\pi}{6}} = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
$= 4\left(\sqrt{3}/2 + i\cdot 1/2\right) = 2\sqrt{3} + 2i$

2.) $\left(e^{i\frac{\pi}{4}}\right)^7 = e^{i\cdot\frac{7\pi}{4}} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$
$= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

3.) $e^{i+\ln 3} = e^{i}e^{\ln 3} = 3(\cos 1 + i\sin 1)$
$= 3\cos 1 + 3i\sin 1$

4.) $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

5.) $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + i(0) = 1$

6.) $e^{4\pi i} = \cos 4\pi + i\sin 4\pi = 1 + i(0) = 1$

7.) $e^{-6\pi i} = \cos(-6\pi) + i\sin(-6\pi) = 1 + i(0) = 1$

Ex: Find all complex numbers z which solve $e^z = 1+i$: Let $z = a+bi$. Then

$e^z = 1+i \Rightarrow e^{a+bi} = 1+i \Rightarrow$

$e^a e^{bi} = 1+i \Rightarrow e^a(\cos b + i\sin b) = 1+i$;

(stop and think) $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$ and

$1+i = \sqrt{2}\cdot\frac{1+i}{\sqrt{2}} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)$, so that

$e^a(\cos b + i\sin b) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right) \Rightarrow e^a = \sqrt{2}$

$\Rightarrow \boxed{a = \ln\sqrt{2}}$ and $\cos b = \frac{1}{\sqrt{2}}$, $\sin b = \frac{1}{\sqrt{2}} \Rightarrow$

$\boxed{b = \frac{\pi}{4} \pm 2\pi n}$ for $n = 0, 1, 2, 3, \ldots$, i.e., $\boxed{z = \ln\sqrt{2} + i\left(\frac{\pi}{4} \pm 2\pi n\right)}$
for $n = 0, 1, 2, 3, \ldots$