Math 21C
Kouba
Exact Change, Differential, Chain Rule

Assume that function $z = f(x, y)$ has continuous partial derivatives and that point (x, y) changes from (x_1, y_1) to (x_2, y_2). Let $z_1 = f(x_1, y_1)$ and $z_2 = f(x_2, y_2)$. Define the *exact change in* f *(or* z*)* to be

$$\Delta f = z_2 - z_1 .$$

Let $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$. Now define the *differential of* f *(or* z*)* to be

$$df = \frac{\partial f}{\partial x}(x_1, y_1)\ \Delta x + \frac{\partial f}{\partial y}(x_1, y_1)\ \Delta y\ .$$

It can be proven using continuity, the Mean Value Theorem, and the differential for a function of one variable that

$$\Delta f = \frac{\partial f}{\partial x}(x_1, y_1)\ \Delta x + \frac{\partial f}{\partial y}(x_1, y_1)\ \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y\ \ ,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. It follows immediately that

$$\Delta f \approx df$$

if both Δx and Δy are"small." Thus, the differential df can be considered an approximation to the exact change Δf.

Now assume that $z = f(x, y)$, $x = g(t)$, and $y = h(t)$. The above equation for Δf leads to the following chain rule :

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\ .$$

If $z = f(x, y)$, $x = g(u, v)$, and $y = h(u, v)$, then the above equation for Δf leads to the following chain rules :

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\ .$$