Math 21C
Kouba
Geometric Series Test

Let r be a real number and consider the geometric series given by

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots .$$

If $r = 1$, then $\displaystyle\sum_{n=0}^{\infty} 1^n = 1+1+1+1+\cdots$ diverges ($= \infty$) by the sequence of partial sums test. If $r = -1$, then $\displaystyle\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+1-1+\cdots$ diverges (oscillation) by the sequence of partial sums test. What happens with other values of r ? Apply the sequence of partial sums test :

$$s_1 = 1,$$
$$s_2 = 1 + r,$$
$$s_3 = 1 + r + r^2,$$
$$s_4 = 1 + r + r^2 + r^3, \cdots$$

and

$$s_n = 1 + r + r^2 + r^3 + \cdots + r^{n-2} + r^{n-1} = \frac{1 - r^{(n-1)+1}}{1-r} = \frac{1-r^n}{1-r} .$$

Then

$$\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-r^n}{1-r}$$

$$= \begin{cases} \dfrac{1-0}{1-r} = \dfrac{1}{1-r} , & \text{if } -1 < r < 1 \\ \textit{Does Not Exist} , & \text{if } r > 1 \text{ or } r < -1 . \end{cases}$$

Thus, the geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r}$$

for $-1 < r < 1$. This series diverges for all other values of r.