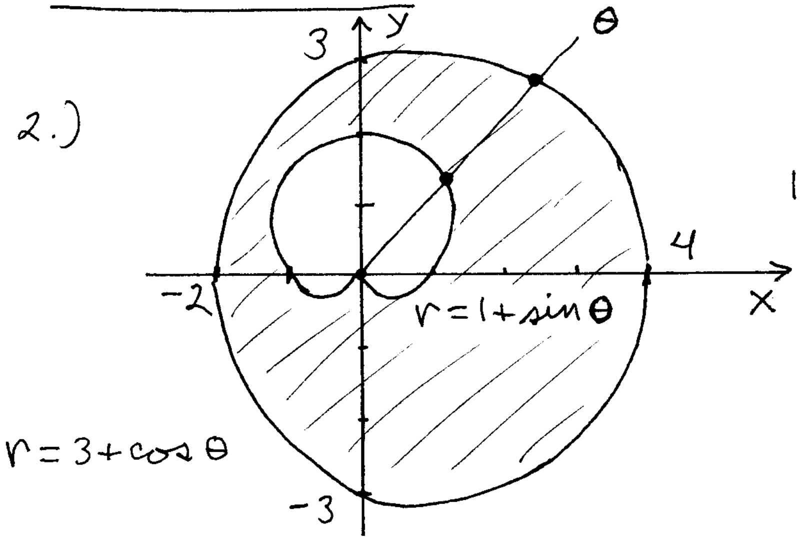
<u>Section 15.4</u>

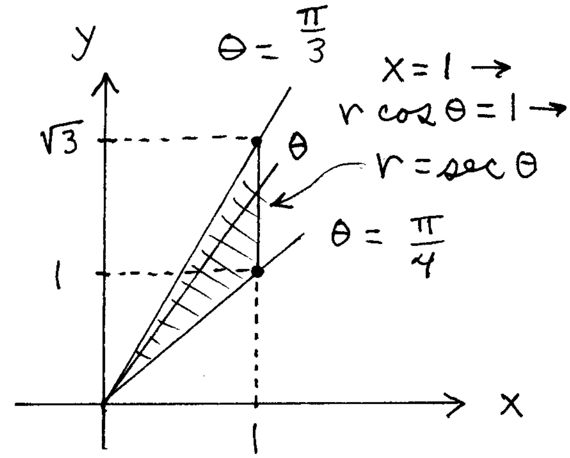
2.)



$0 \le \theta \le 2\pi$

and

$1+\sin\theta \le r \le 3+\cos\theta$



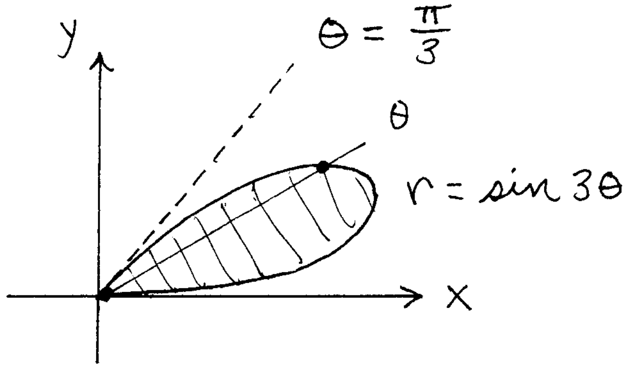
3.) $\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$

and

$0 \le r \le \sec\theta$

5.) $\frac{\pi}{3} \le \theta \le \frac{5\pi}{3}$ and $0 \le r \le 5$

6.)



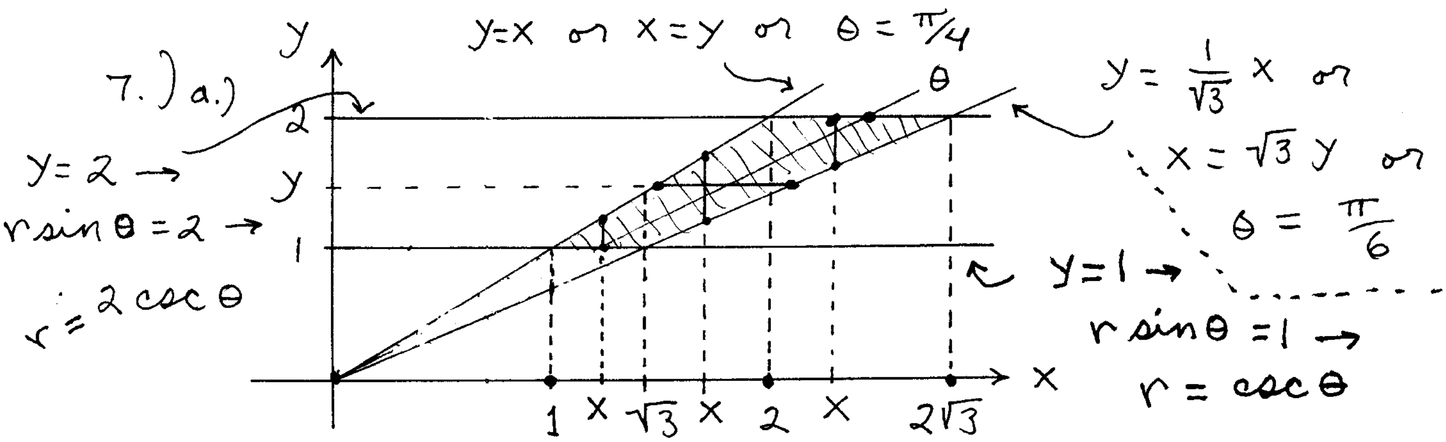
$\sin 3\theta = 0 \rightarrow$

$3\theta = 0 \text{ or } \pi \rightarrow$

$\theta = 0 \text{ or } \frac{\pi}{3}$

$0 \le \theta \le \frac{\pi}{3}$ and $0 \le r \le \sin 3\theta$

7.) a.)

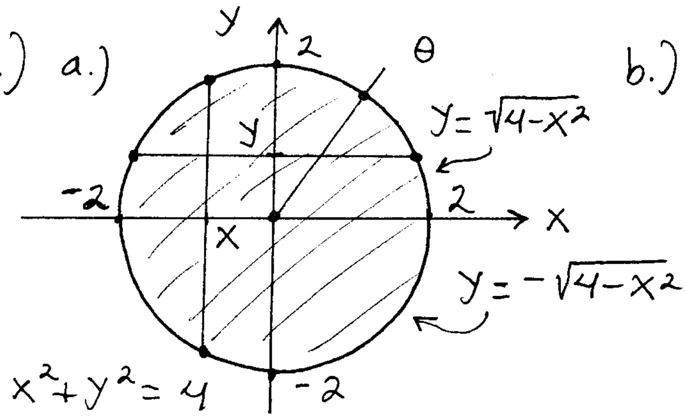


b.) $1 \le y \le 2$ and $y \le x \le \sqrt{3}\, y$

c.) $1 \le x \le \sqrt{3}$ and $1 \le y \le x$, $\sqrt{3} \le x \le 2$ and $\frac{1}{\sqrt{3}} x \le y \le x$,
$2 \le x \le 2\sqrt{3}$ and $\frac{1}{\sqrt{3}} x \le y \le 2$

d.) $\frac{\pi}{6} \le \theta \le \frac{\pi}{4}$ and $\csc\theta \le r \le 2\csc\theta$

8.) a.)



b.) $-2 \le x \le 2$ and
$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

c.) $0 \le \theta \le 2\pi$
and
$0 \le r \le 2$

13.) $$\iint_R r^2\, dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^{1+\cos\theta} d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}(1+\cos\theta)^4 d\theta = \frac{1}{4}\int_0^{2\pi} (\cos^4\theta + 4\cos^3\theta + 6\cos^2\theta + 4\cos\theta + 1)\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left(\left(\frac{1}{2}(1+\cos 2\theta)\right)^2 + 4\cos\theta(1-\sin^2\theta) + 6\cdot\frac{1}{2}(1+\cos 2\theta) + 4\cos\theta + 1\right) d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\Big(\frac{1}{4}\Big(1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\Big) + 4\cos\theta - 4\cos\theta\sin^2\theta$$
$$+ 3 + 3\cos 2\theta + 4\cos\theta + 1\Big)\, d\theta$$

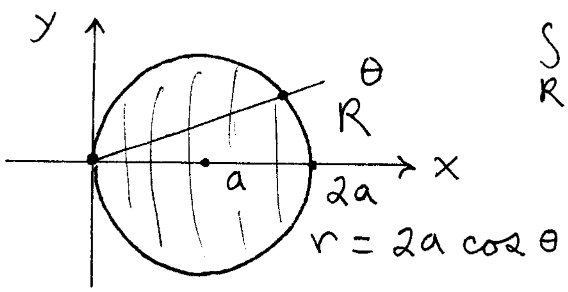
$$= \frac{1}{4}\int_0^{2\pi}\left(\left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right) + 8\cos\theta + 3\cos 2\theta - 4\cos\theta\sin^2\theta + 4\right) d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left(\frac{35}{8} + \frac{7}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta + 8\cos\theta - 4\cos\theta\sin^2\theta\right) d\theta$$

$$= \frac{1}{4}\left(\frac{35}{8}\theta + \frac{7}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + 8\sin\theta - \frac{4}{3}\sin^3\theta\right)\Big|_0^{2\pi}$$

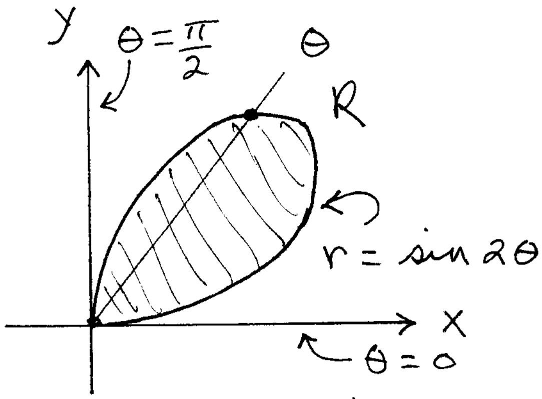
$$= \frac{1}{4}\left(\frac{35}{8}\right)2\pi = \frac{35}{16}\pi.$$

16.)



$$\iint_R y^2\, dA$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} (r\sin\theta)^2\, r\, dr\, d\theta$$

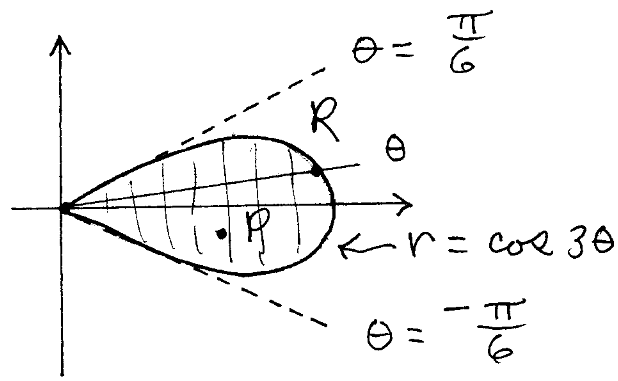
18.)



$\sin 2\theta = 0 \rightarrow$
$2\theta = 0$ or $\pi \rightarrow$
$\theta = 0$ or $\frac{\pi}{2}$

$$\int_R y^2 \, dA = \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} (r \sin \theta)^2 \, r \, dr \, d\theta$$

20.)



$\cos 3\theta = 0 \rightarrow$
$3\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2} \rightarrow$
$\theta = \frac{\pi}{6}$ or $-\frac{\pi}{6}$;

density is $\delta(P) = \delta(x, y) = xy$:

$$\text{Mass} = \int_R \delta(P) \, dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} (r \cos \theta)(r \sin \theta) \, r \, dr \, d\theta \, ;$$

the center of mass $(\bar{x}, \bar{y})$ is given by

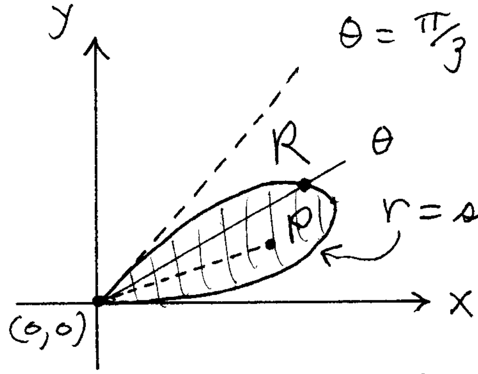
$$\bar{x} = \frac{\int_R x \, \delta(P) \, dA}{\text{Mass}} = \frac{\int_R x(xy) \, dA}{\text{Mass}} = \frac{\int_R x^2 y \, dA}{\text{Mass}} \rightarrow$$

$$\bar{x} = \frac{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} (r \cos \theta)^2 (r \sin \theta) \, dr \, d\theta}{\text{Mass}} \, ;$$

$$\bar{y} = \frac{\int_R y \, \delta(P) \, dA}{\text{Mass}} = \frac{\int_R y(xy) \, dA}{\text{Mass}} = \frac{\int_R xy^2 \, dA}{\text{Mass}} \rightarrow$$

$$\bar{y} = \frac{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} (r\cos\theta)(r\sin\theta)^2 \cdot r \, dr \, d\theta}{\text{Mass}}$$

21.)



$\sin 3\theta = 0 \rightarrow$

$3\theta = 0$ or $\pi \rightarrow$

$\theta = 0$ or $\frac{\pi}{3}$;

$f(P)$ is the distance from $P = (x, y)$ to $(0,0)$

so $f(P) = \sqrt{x^2+y^2}$;

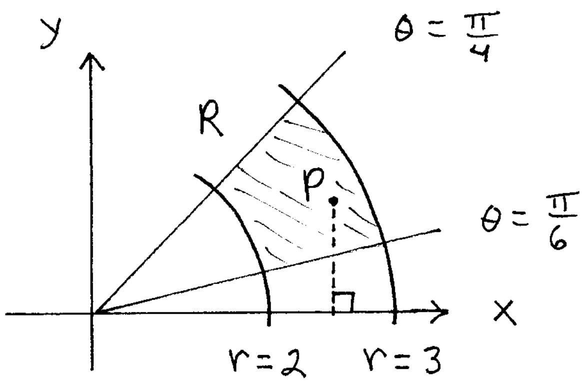
Area of $R = \int_R 1 \, dA = \int_0^{\pi/3} \int_0^{\sin 3\theta} r \, dr \, d\theta$;

$$\text{AVE} = \frac{1}{\text{area } R} \int_R f(P) \, dA = \frac{1}{\text{area } R} \int_R \sqrt{x^2+y^2} \, dA$$

$$= \frac{1}{\text{area } R} \int_0^{\frac{\pi}{3}} \int_0^{\sin 3\theta} \sqrt{r^2} \cdot r \, dr \, d\theta$$

$$= \frac{1}{\text{area } R} \int_0^{\frac{\pi}{3}} \int_0^{\sin 3\theta} r^2 \, dr \, d\theta$$

22.)



$f(P) = f(x, y) = y$;

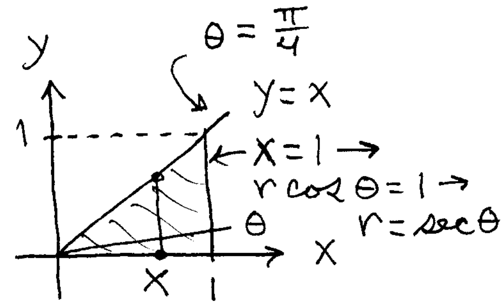
Area of $R = \int_R 1 \, dA$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_2^3 r \, dr \, d\theta$;

$$\text{AVE} = \frac{1}{\text{area } R} \int_R f(P) \, dA = \frac{1}{\text{area } R} \int_R y \, dA$$

$$= \frac{1}{\text{area } R} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_2^3 (r\sin\theta) \cdot r \, dr \, d\theta \, .$$

25.) $\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dy\, dx$



$= \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \sqrt{r^2} \cdot r \, dr\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{3} r^3 \Big|_{r=0}^{r=\sec\theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3}\sec^3\theta \, d\theta$

$= \frac{1}{3} \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$ $\quad \Big(\int \sec^3\theta\, d\theta = \int \sec\theta \sec^2\theta \, d\theta$

(Let $u = \sec\theta$, $dv = \sec^2\theta \, d\theta$

$du = \sec\theta \tan\theta \, d\theta$, $v = \tan\theta$)

$= \sec\theta\tan\theta - \int \sec\theta \tan^2\theta \, d\theta$

$= \sec\theta\tan\theta - \int \sec\theta (\sec^2\theta - 1)\, d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta \, d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \ln|\sec\theta + \tan\theta| \rightarrow$

$2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \rightarrow$

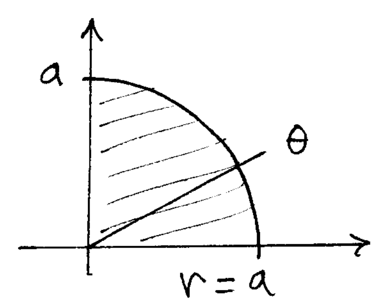
$\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)$. $\Big)$

$= \frac{1}{3}\cdot\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{6}\left((\sqrt{2})\cdot(1) + \ln(\sqrt{2}+1)\right) - \frac{1}{6}(0 + \cancel{\ln 1}^{\,0})$

$= \frac{1}{6}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$

29.) a.) $\int_R \cos(x^2+y^2)\, dA$

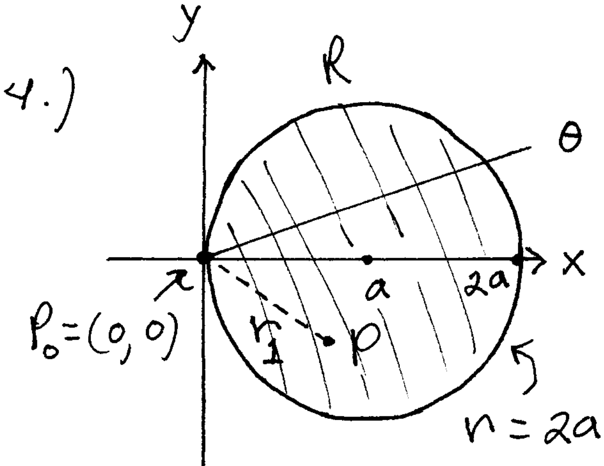


$= \int_0^{\frac{\pi}{2}} \int_0^a \cos(r^2)\cdot r \, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(r^2)\Big|_0^a d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin a^2 \, d\theta$

$= \left(\frac{1}{2}\sin a^2\right)\theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}\sin a^2$.

34.)



area of $R = \pi a^2$; density

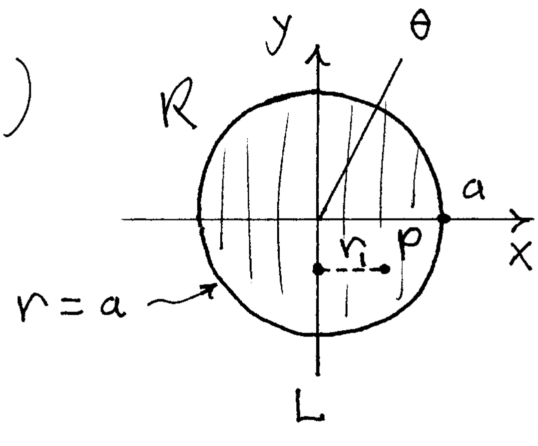
$$f(P) = \frac{\text{mass}}{\text{area}} = \frac{M}{\pi a^2} ;$$

the distance from $P_0=(0,0)$ to $P=(x,y)$ is $r_1 = \sqrt{x^2+y^2}$; then

$$\text{M. of I.} = \int_R r_1^2 f(P)\, dA = \int_R (x^2+y^2) f(P)\, dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} (r^2) \frac{M}{\pi a^2} \cdot r\, dr\, d\theta$$

35.)



area of $R = \pi a^2$; density

$$f(P) = \frac{M}{\pi a^2} ;$$

the distance from pt. $P=(x,y)$ to line L is $r_1 = |x|$; then

$$\text{M. of I.} = \int_R r_1^2 f(P)\, dA = \int_R x^2 f(P)\, dA$$

$$= \int_0^{2\pi} \int_0^a (r\cos\theta)^2 \cdot \frac{M}{\pi a^2} \cdot r\, dr\, d\theta .$$

39.) a.) $\int_{R_1} f(P)\,dA = \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} \cdot r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \left(\frac{-1}{2} e^{-r^2} \Big|_0^a \right) d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{-1}{2} e^{-a^2} - \frac{-1}{2} e^0 \right) d\theta$

$= \left(\frac{1}{2} - \frac{1}{2} e^{-a^2} \right) \theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}\left(1 - e^{-a^2}\right)$;

$\int_{R_3} f(P)\,dA = \int_0^{\frac{\pi}{2}} \int_0^{a\sqrt{2}} e^{-r^2} \cdot r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \left(\frac{-1}{2} e^{-r^2} \Big|_0^{a\sqrt{2}} \right) d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{-1}{2} e^{-2a^2} - \frac{-1}{2} e^0 \right) d\theta$

$= \left(\frac{1}{2} - \frac{1}{2} e^{-2a^2} \right) \theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}\left(1 - e^{-2a^2}\right)$.

b.) $\int_{R_2} f(P)\,dA = \int_0^a \int_0^a e^{-x^2 - y^2}\,dy\,dx$

$= \int_0^a \int_0^a e^{-x^2} \cdot e^{-y^2}\,dy\,dx$

$= \int_0^a e^{-x^2} \left(\int_0^a e^{-y^2}\,dy \right) dx$

$= \left(\int_0^a e^{-y^2}\,dy \right) \left(\int_0^a e^{-x^2}\,dx \right)$

$= \left(\int_0^a e^{-x^2}\,dx \right) \left(\int_0^a e^{-x^2}\,dx \right)$

$= \left(\int_0^a e^{-x^2}\,dx \right)^2$; since

$\int_{R_1} f(P)\,dA \le \int_{R_2} f(P)\,dA \le \int_{R_3} f(P)\,dA \rightarrow$

$$\frac{\pi}{4}(1-e^{-a^2}) \le \left(\int_0^a e^{-x^2}dx\right)^2 \le \frac{\pi}{4}(1-e^{-2a^2})$$

c.) Since $\lim_{a\to\infty} \frac{\pi}{4}(1-e^{-a^2}) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$

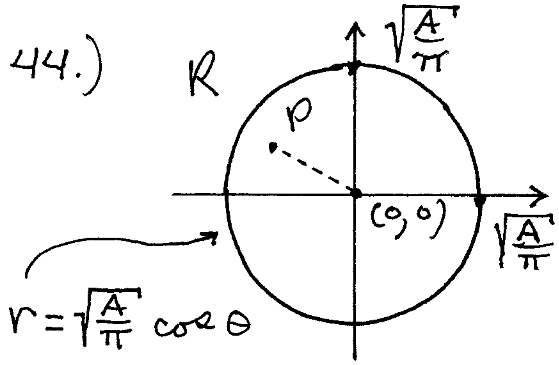
and $\lim_{a\to\infty} \frac{\pi}{4}(1-e^{-2a^2}) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$,

it follows from the Squeeze Principle that

$$\lim_{a\to\infty} \left(\int_0^a e^{-x^2}dx\right)^2 = \frac{\pi}{4} \rightarrow$$

$$\int_0^a e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

44.)



Circle of area $A \rightarrow$ $A = \pi r^2 \rightarrow r = \sqrt{\frac{A}{\pi}}$; $f(P)$ is the distance from pt. $P=(x,y)$ to $(0,0)$, i.e., $f(P) = \sqrt{x^2+y^2}$; the average value of f over region R is

$$AVE = \frac{1}{\text{area } R} \iint_R f(P)\, dA = \frac{1}{A}\iint_R \sqrt{x^2+y^2}\, dA$$

$$= \frac{1}{A}\int_0^{2\pi}\int_0^{\sqrt{\frac{A}{\pi}}} \sqrt{r^2}\cdot r\, dr\, d\theta$$

$$= \frac{1}{A}\int_0^{2\pi}\int_0^{\sqrt{\frac{A}{\pi}}} r^2\, dr\, d\theta .$$