Section 10.4

1.) $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$; let $f(x) = \frac{1}{x^{1.1}}$ which is positive, decreasing, and continuous for $x \geq 1$; then

$$\int_1^{\infty} \frac{1}{x^{1.1}}\,dx = \lim_{A\to\infty} \int_1^A x^{-1.1}\,dx = \lim_{A\to\infty} \left.\frac{x^{-0.1}}{-0.1}\right|_1^A$$

$$= \lim_{A\to\infty} \left.\frac{-10}{x^{0.1}}\right|_1^A = \lim_{A\to\infty} \left(\frac{-10}{A^{0.1}} - \frac{-10}{1}\right) = 0 + 10 = 10 < \infty.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges.

3.) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$; let $f(x) = \frac{x}{x^2+1}$ which is positive and continuous for $x \geq 1$; check decreasing : $f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$+ \quad 0 \quad -$

$x=1$ $\quad f'$, so f is ↓ for $x \geq 1$; then

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{A\to\infty} \left.\frac{1}{2}\ln(x^2+1)\right|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln(A^2+1) - \frac{1}{2}\ln 2\right) = \infty - \frac{1}{2}\ln 2 = \infty,$$

so $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

4.) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$; let $f(x) = \frac{1}{x^2+1}$ which is positive, decreasing, and continuous for $x \geq 1$; then

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{A\to\infty} \arctan x \Big|_1^A$$

$$= \lim_{A\to\infty} (\arctan A - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty,$$

so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

5.) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$; let $f(x) = \frac{1}{x \ln x}$ which is positive, decreasing, and continuous for $x \geq 2$; then

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{A \to \infty} \ln|\ln x| \Big|_2^A = \lim_{A \to \infty} \left(\ln|\ln A| - \ln|\ln 2| \right)$$

$= \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

6.) $\sum_{n=1}^{\infty} \frac{1}{n+1000}$; let $f(x) = \frac{1}{x+1000}$ which is positive, decreasing, and continuous for $x \geq 1$; then

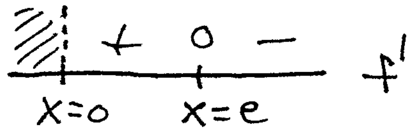
$$\int_1^{\infty} \frac{1}{x+1000}\,dx = \lim_{A \to \infty} \ln|x+1000| \Big|_1^A$$

$$= \lim_{A \to \infty} \left(\ln|A+1000| - \ln 1001 \right) = \infty,$$ so

$\sum_{n=1}^{\infty} \frac{1}{n+1000}$ diverges.

7.) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$; let $f(x) = \frac{\ln x}{x}$, which is positive (for $x \geq 2$!) and continuous; is f decreasing? check:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$



so f is decreasing for $x \geq 3$! then

$$\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{A \to \infty} \frac{1}{2} (\ln x)^2 \Big|_3^A$$

$$= \lim_{A \to \infty} \left(\frac{1}{2} (\ln A)^2 - \frac{1}{2} (\ln 3)^2 \right) = \infty,$$ so

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

10.) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-series test since $p = 3 > 1$.

12.) $\sum_{n=1}^{\infty} \frac{1}{n^{0.999}}$ diverges by the p-series test since $p = 0.999 \leq 1$.

13.) a.) Consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 1$; let $f(x) = \frac{1}{x^p}$, which is positive, decreasing, and continuous for $x \geq 1$; then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{A \to \infty} \int_1^A x^{-p}\,dx = \lim_{A \to \infty} \frac{x^{1-p}}{1-p} \Big|_1^A$$

$$= \lim_{A \to \infty} \frac{1}{1-p}\left(\frac{1}{\underset{\nwarrow\, p-1>0}{A^{p-1}}} - 1\right) = \frac{1}{1-p}(0-1) = \frac{1}{p-1} < \infty,$$

so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

b.) From equation (*) in class handout:

$$\int_1^{n+1} \frac{1}{x^p}\,dx < \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} < \frac{1}{1^p} + \int_1^n \frac{1}{x^p}\,dx \Rightarrow$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx < \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots < 1 + \int_1^{\infty} \frac{1}{x^p}\,dx \Rightarrow$$

$$\frac{1}{p-1} < \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots < 1 + \frac{1}{p-1} = \frac{p}{p-1}$$

14.) Consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $0 < p < 1$; let $f(x) = \frac{1}{x^p}$, which is positive, continuous, and decreasing; then

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{A\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^A$$

$$= \lim_{A\to\infty} \frac{1}{1-p}\left(\underset{\uparrow\ 1-p>0}{A^{1-p}}-1\right) = \infty,$$ so

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0<p<1$;

if $p=1$, then $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by harmonic series test ; if $p=0$, then $\sum_{n=1}^{\infty}\frac{1}{n^0} = \sum_{n=1}^{\infty} 1$ diverges by nth term test ; if $p<0$, say $p=-r<0$, then $\sum_{n=1}^{\infty}\frac{1}{n^{-r}} = \sum_{n=1}^{\infty} n^r$ diverges by nth term test. Thus, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for $p\le 1$.

16.) $\sum_{n=1}^{\infty}\frac{1}{n^4}$; let $f(x)=\frac{1}{x^4}$

a.) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4} \approx 1.0788 = S_4$

b.) By equation (*)(*) on class handout :

$$\int_{4+1}^\infty \frac{1}{x^4}\,dx < \underbrace{\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}+\cdots}_{R_4} < \int_4^\infty \frac{1}{x^4}\,dx \Rightarrow$$

$$\lim_{A\to\infty} \frac{-1}{3x^3}\Big|_5^A < R_4 < \lim_{A\to\infty}\frac{-1}{3x^3}\Big|_4^A \Rightarrow$$

$$\lim_{A\to\infty}\left(\overset{0}{\cancel{\frac{-1}{3A^3}}}-\frac{-1}{3\cdot 5^3}\right) < R_4 < \lim_{A\to\infty}\left(\overset{0}{\cancel{\frac{-1}{3A^3}}}-\frac{-1}{3\cdot 4^3}\right) \Rightarrow$$

$0.0027 < R_4 < 0.0052$. Then

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = S_4 + R_4 = 1.0788 + R_4 \Rightarrow$$

c.) $\underline{\underline{1.0815}} = 1.0788 + 0.0027 < \sum_{n=1}^{\infty}\frac{1}{n^4} < 1.0788 + 0.0052 = \underline{\underline{1.0840}}$

18.) $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$; let $f(x)=\frac{1}{x^2+x}=\frac{1}{x}-\frac{1}{x+1}$

a.) $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20} = 0.8000 = S_4$

b.) By equation (*)(*) :

$$\int_{4+1}^{\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx < \underbrace{\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\cdots}_{R_4} < \int_{4}^{\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx \Rightarrow$$

$$\left(\ln x-\ln(x+1)\right)\Big|_5^{\infty} < R_4 < \left(\ln x-\ln(x+1)\right)\Big|_4^{\infty} \Rightarrow$$

$$\lim_{A\to\infty}\ln\left(\frac{x}{x+1}\right)\Big|_5^{A} < R_4 < \lim_{A\to\infty}\ln\left(\frac{x}{x+1}\right)\Big|_4^{A} \Rightarrow$$

$$\lim_{A\to\infty}\ln\left(\frac{A}{A+1}\right)-\ln\left(\frac{5}{6}\right) < R_4 < \lim_{A\to\infty}\ln\left(\frac{A}{A+1}\right)-\ln\left(\frac{4}{5}\right) \Rightarrow$$

$$\overset{0}{\cancel{\ln 1}}-\ln\left(\frac{5}{6}\right) < R_4 < \overset{0}{\cancel{\ln 1}}-\ln\left(\frac{4}{5}\right) \Rightarrow$$

$0.1823 < R_4 < 0.2231$. Then

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = S_4 + R_4 = 0.8 + R_4 \Rightarrow$$

c.) $0.9823 < \sum_{n=1}^{\infty}\frac{1}{n^4} < 1.0231$

20.) $\sum_{n=1}^{\infty}\frac{1}{n^3} = \underbrace{\frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{1000^3}}_{S_{1000}} + \underbrace{\frac{1}{1001^3}+\frac{1}{1002^3}+\cdots}_{R_{1000}}$;

a.)

by equation (*)(*)

$$\int_{1000+1}^{\infty} \frac{1}{x^3}\,dx < R_{1000} < \int_{1000}^{\infty} \frac{1}{x^3}\,dx \Rightarrow$$

$$\lim_{A\to\infty} \frac{-1}{2x^2}\Big|_{1001}^{A} < R_{1000} < \lim_{A\to\infty} \frac{-1}{2x^2}\Big|_{1000}^{A} \Rightarrow$$

$$\lim_{A\to\infty}\left(\cancelto{0}{\frac{-1}{2A^2}} - \frac{-1}{2(1001)^2}\right) < R_{1000} < \lim_{A\to\infty}\left(\cancelto{0}{\frac{-1}{2A^2}} - \frac{-1}{2(1000)^2}\right) \Rightarrow$$

$$0.00000049 < R_{1000} < 0.0000005$$

b.) $R_n < \int_{n}^{\infty} \frac{1}{x^3}\,dx = \lim_{n\to\infty} \frac{-1}{2x^2}\Big|_{n}^{A} = \lim_{n\to\infty}\left(\cancelto{0}{\frac{-1}{2A^2}} - \frac{-1}{2n^2}\right)$

$$= \frac{1}{2n^2} < 0.0001 \Rightarrow n^2 > \frac{1}{2(0.0001)} \Rightarrow$$

$n > \sqrt{\frac{1}{0.0002}} \approx 70.7$ so choose $n \geq 71$.

21.) a.) $\sum_{n=1}^{\infty} \frac{1}{n^4}$ then $R_n < \int_{n}^{\infty} \frac{1}{x^4}\,dx = \lim_{A\to\infty} \frac{-1}{3x^3}\Big|_{n}^{A}$

$$= \lim_{A\to\infty}\left(\cancelto{0}{\frac{-1}{3A^3}} - \frac{-1}{3n^3}\right) = \frac{1}{3n^3} < 0.0001 \Rightarrow n^3 > \frac{1}{0.0003} \Rightarrow$$

$n > \sqrt[3]{\frac{1}{0.0003}} \approx 14.9$ so choose $n \geq 15$.

b.) $s_{15} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{15^4} \approx 1.0822$ estimates the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ with error at most $R_{15} < 0.0001$.

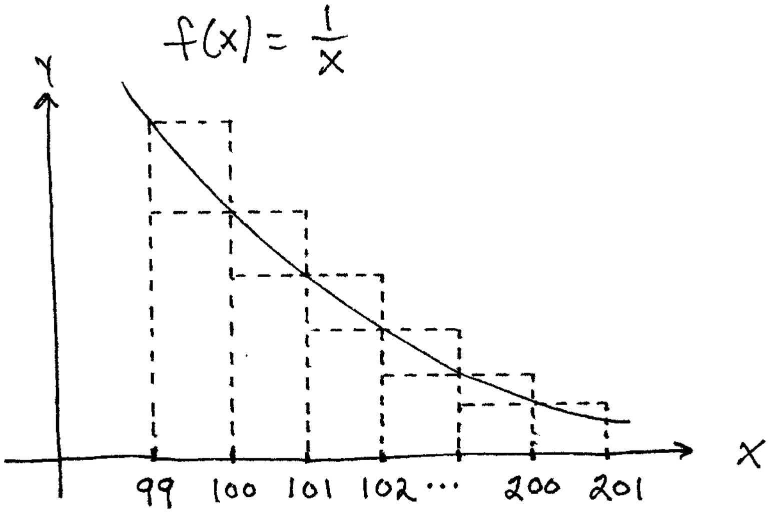
26.) Let $f(x) = \frac{1}{\sqrt{x}}$ and use equation $\ast$):

$$\int_{1}^{n+1} \frac{1}{\sqrt{x}}\,dx < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} < f(1) + \int_{1}^{n} \frac{1}{\sqrt{x}}\,dx \Rightarrow$$

$$2\sqrt{x}\Big|_{1}^{n+1} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} < 1 + 2\sqrt{x}\Big|_{1}^{n} \Rightarrow$$

$$2\sqrt{n+1} - 2 < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} < 1 + 2\sqrt{n} - 2 = 2\sqrt{n} - 1 .$$

27.) a.)



By using integrals and rectangles:

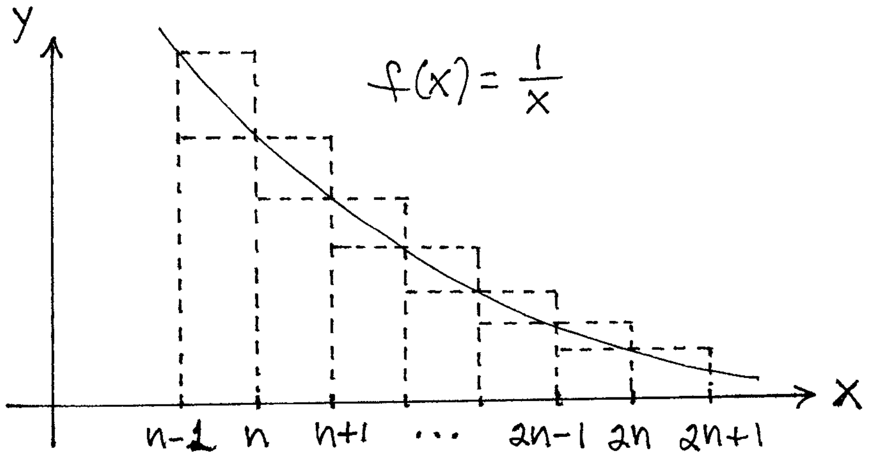
$$\int_{100}^{201} \frac{1}{x}\,dx < \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} < \int_{99}^{200} \frac{1}{x}\,dx \Rightarrow$$

$$\ln x \Big|_{100}^{201} < \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} < \ln x \Big|_{99}^{200} \Rightarrow$$

$$\ln 201 - \ln 100 < \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} < \ln 200 - \ln 99 \Rightarrow$$

$$\ln \frac{201}{100} < \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} < \ln \frac{200}{99} .$$

b.)



$$\int_{n}^{2n+1} \frac{1}{x}\,dx < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < \int_{n-1}^{2n} \frac{1}{x}\,dx \Rightarrow$$

$$\ln x \Big|_{n}^{2n+1} < \sum_{i=n}^{2n} \frac{1}{i} < \ln x \Big|_{n-1}^{2n} \Rightarrow$$

$$\ln(2n+1) - \ln n < \sum_{i=n}^{2n} \frac{1}{i} < \ln 2n - \ln(n-1) \Rightarrow$$

$$\ln\left(\frac{2n+1}{n}\right) < \sum_{i=n}^{2n} \frac{1}{i} < \ln\left(\frac{2n}{n-1}\right) \Rightarrow$$

$$\lim_{n\to\infty} \ln\left(\frac{2n+1}{n}\right) \leq \lim_{n\to\infty} \sum_{i=n}^{2n} \frac{1}{i} \leq \lim_{n\to\infty} \ln\left(\frac{2n}{n-1}\right) \Rightarrow$$

$$\ln 2 \leq \lim_{n\to\infty} \sum_{i=n}^{2n} \frac{1}{i} \leq \ln 2 \Rightarrow$$

$$\lim_{n\to\infty} \sum_{i=n}^{2n} \frac{1}{i} = \ln 2 \quad .$$

28.) b.) $\int_1^\infty \frac{1}{x^2}\,dx < \sum_{n=1}^{\infty} \frac{1}{n^2} < f(1) + \int_1^\infty \frac{1}{x^2}\,dx \Rightarrow$

$$\lim_{\to\infty} \left.\frac{-1}{x}\right|_1^A < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \lim_{A\to\infty} \left.\frac{-1}{x}\right|_1^A \Rightarrow$$

$$\lim_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \lim_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{1}\right) \Rightarrow$$

$$1 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + 1 = 2 \quad .$$