Section 10.5

1.) $\frac{1}{n^2+3} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p=2>1)$, so $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges.

2.) $\frac{n+2}{(n+1)\sqrt{n}} \ge \frac{n+2}{(n+2)\sqrt{n}} = \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series test $(p=\frac{1}{2}\le 1)$, so $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n}}$

3.) $\frac{\sin^2 n}{n^2} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p= \quad >1)$, so $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ converges.

4.) $\frac{1}{n2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges by geometric series test $(r=\frac{1}{2}, -1<r<1)$, so $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges!

5.) $$\lim_{n\to\infty} \frac{\frac{5n+1}{(n+2)n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n+1}{n^3+2n^2} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{5n^3+n^2}{n^3+2n^2} = \lim_{n\to\infty} \frac{5+\frac{1}{n}}{1+\frac{2}{n}} = \frac{5+0}{1+0} = 5 > 0;$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p=2>1)$, then $\sum_{n=1}^{\infty} \frac{5n+1}{(n+2)n^2}$ converges.

6.) $\lim_{n\to\infty} \dfrac{\frac{2^n+n}{3^n}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{2^n+n}{\cancel{3^n}} \cdot \frac{\cancel{3^n}}{2^n}$

$= \lim_{n\to\infty} \left(1+\frac{n}{2^n}\right) = 1 + \lim_{n\to\infty} \frac{n}{2^n}$

$= 1 + \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 1 + \frac{1}{\infty} = 1+0 = 1 > 0$;

since $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges

by geometric series test $(r = 2/3,$ $-1<r<1)$, then $\sum_{n=1}^{\infty} \frac{2^n+n}{3^n}$ converges

7.) $\lim_{n\to\infty} \dfrac{\frac{n+1}{(5n+2)\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n+1}{(5n+2)\cancel{\sqrt{n}}} \cdot \frac{\cancel{\sqrt{n}}}{1}$

$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{5+\frac{2}{n}} = \frac{1+0}{5+0} = \frac{1}{5} > 0$;

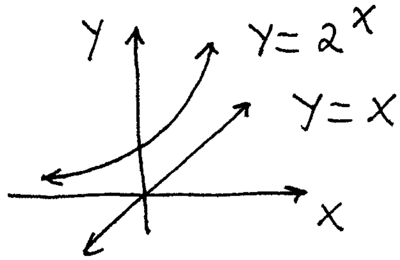
since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series test $(p = \frac{1}{2} \leq 1)$, then $\sum_{n=1}^{\infty} \frac{n+1}{(5n+2)\sqrt{n}}$

diverges.

8.) $$\lim_{n\to\infty} \frac{\frac{\left(1+\frac{1}{n}\right)^n}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\cancel{n^2}} \cdot \frac{\cancel{n^2}}{1} = e > 0;$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$, so $\sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n}{n^2}$ converges.

10.)



$2^x > x$ so

$\frac{2^n}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$, and

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series),

so $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges by the comparison test.

11.) $\frac{1}{n^n} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$, so $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the comparison test.

13.) $$\lim_{n\to\infty} \frac{\frac{4n+1}{(2n+3)n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{4n+1}{(2n+3)\cancel{n^2}} \cdot \frac{\cancel{n^2}}{1} = \frac{4}{2} = 2 > 0;$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$,

so $\sum_{n=1}^{\infty} \frac{4n+1}{(2n+3)n^2}$ converges by limit comparison test.

15.) $\frac{1+\cos n}{n^2} \le \frac{1+1}{n^2} = \frac{2}{n^2}$ and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a convergent p-series ($p=2>1$), so $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ converges by comparison test.

16.) See problem 7 in Section 10.4.

17.) $\lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1}$

$= \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{2\sqrt{n}}{1}$

$= \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$; since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test ($p = \frac{3}{2} > 1$), then $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by the limit comparison test.

18.) $\sum_{n=1}^{\infty} \frac{5^n}{n^n} = \frac{5^1}{1^1} + \frac{5^2}{2^2} + \frac{5^3}{3^3} + \frac{5^4}{4^4} + \frac{5^5}{5^5} + \frac{5^6}{\underline{\underline{6^6}}} + \frac{5^7}{7^7} + \frac{5^8}{8^8} + \cdots$, then it follows that

$\frac{5^n}{n^n} \le \frac{5^n}{6^n}$ for $n \ge 6$; and $\sum_{n=6}^{\infty} \frac{5^n}{6^n} = \sum_{n=6}^{\infty} \left(\frac{5}{6}\right)^n$

is a convergent geometric series ($r = 5/6$ and $-1 < r < 1$), so that $\sum_{n=1}^{\infty} \frac{5^n}{n^n}$ converges by the comparison test.

20.) $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\ln n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty;$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$ diverges by limit comparison test.

21.) $\sum_{n=1}^{\infty} \frac{e^n}{\pi^n} = \sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ which converges by the geometric series test since $r = \frac{e}{\pi}$ and $-1 < r < 1$.

23.) $\lim_{k\to\infty} \frac{3k+1}{2k+10} = \lim_{k\to\infty} \frac{3+\frac{1}{k}}{2+\frac{10}{k}} = \frac{3}{2} \neq 0$ so $\sum_{k=1}^{\infty} \frac{3k+1}{2k+10}$ diverges by the nth term test.

24.) $\lim_{k\to\infty} \frac{\frac{4}{2k^2-k}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{4}{2k^2-k} \cdot \frac{k^2}{1} = \lim_{k\to\infty} \frac{4}{2-\frac{1}{k}}$

$= \frac{4}{2-0} = 2 > 0$; since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-series test $(p=2>1)$, then $\sum_{k=1}^{\infty} \frac{4}{2k^2-k}$ converges by the limit comparison test.

25.) $\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty;$

since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series), then $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the limit comparison test.

26.) $\lim_{n\to\infty} \csc\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{\sin\left(\frac{1}{n}\right)} = \frac{1}{\sin(0)} = \frac{1}{0} = \pm\infty$

$\neq 0$, so $\sum_{n=1}^{\infty} \csc(1/n)$ diverges by the nth term test.

27.) $\lim_{n\to\infty} \left(\frac{n+1}{n+3}\right)^n = \lim_{n\to\infty} \left(\frac{n+3-2}{n+3}\right)^n = \lim_{n\to\infty} \left(1+\frac{-2}{n+3}\right)^n$

$= \lim_{n\to\infty} \left(1+\frac{1}{\frac{-(n+3)}{2}}\right)^{\frac{-(n+3)}{2}\cdot\frac{-2}{n+3}\cdot n}$

$= \lim_{n\to\infty} \left[\left(1+\frac{1}{\frac{-(n+3)}{2}}\right)^{\frac{-(n+3)}{2}}\right]^{\frac{-2n}{n+3}} = [e]^{-2} \neq 0,$

so $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+3}\right)^n$ diverges by the nth term test.

28.) Consider $\left\{\frac{n}{2n-1}\right\}$: $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{6}, \frac{5}{8}, \frac{6}{11}, \ldots$ then $\frac{n}{2n-1} < \frac{3}{4}$ iff $4n < 6n-3$ iff $3 < 2n$ for $n \geq 2$, thus $\left(\frac{n}{2n-1}\right)^n \leq \left(\frac{3}{4}\right)^n$ for $n \geq 2$, so that $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$ is a convergent geometric series since $r = \frac{3}{4}$ and $-1 < r < 1$; thus $\sum_{n=2}^{\infty} \left(\frac{n}{2n-1}\right)^n$ converges by the comparison test.

29.) $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n = \infty$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$; then <u>no conclusion</u> can be made about $\sum_{n=1}^{\infty} a_n$: let $b_n = \frac{1}{\sqrt{n}}$ then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ by p-series test $(p = \frac{1}{2} \leq 1)$; let $a_n = \frac{1}{n}$ then $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (harmonic series) and $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{\sqrt{n}}{1} = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$; let $a_n = \frac{1}{n^2}$ then $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$ and $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{n^2} \cdot \frac{\sqrt{n}}{1}$ $= \lim_{n \to \infty} \frac{1}{n^{3/2}} = 0$.

30.) $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n$ converges, and $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$; let $b_n = \frac{1}{n^3}$ then $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges; if $a_n = \frac{1}{n^2}$ then $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges; if $a_n = \frac{1}{n}$ then $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, thus <u>no conclusion</u> can be made about $\sum_{n=1}^{\infty} a_n$.

31.) $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n$ converges, $3b_n \leq a_n \leq 5b_n$; since $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} 5b_n$ converges ; since $0 \leq a_n \leq 5b_n$ and $\sum_{n=1}^{\infty} 5b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ <u>converges</u> by the comparison test.

32.) $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n = \infty$, and $3b_n \leq a_n \leq 5b_n$;

$\sum_{n=1}^{\infty} 3b_n$ diverges since $\sum_{n=1}^{\infty} b_n$ diverges ; since $0 \le 3b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges by the comparison test.

33.) $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n$ converges, and $a_n < b_n^2$; since $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} b_n = 0$, so that there is some integer N such that if $n > N$, then $b_n < 1$; if $b_n < 1$, then $b_n^2 < b_n$ so $\sum_{n=N}^{\infty} b_n^2$ converges since $\sum_{n=N}^{\infty} b_n$ converges. Since $0 \le a_n < b_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ <u>converges</u> by the comparison test.

34.) $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n = \infty$, $\lim_{n\to\infty} b_n = 0$, and $a_n < b_n^2$; let $b_n = \frac{1}{\sqrt{n}}$ then $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} b_n = \infty$; let $a_n = \frac{1}{2n}$, then $a_n < b_n^2 = \frac{1}{n}$ and $\sum_{n=1}^{\infty} a_n$ diverges ; let $a_n = \frac{1}{n^2}$, then $a_n < b_n^2$ and $\sum_{n=1}^{\infty} a_n$ converges ; thus, <u>no conclusion</u> can be made about $\sum_{n=1}^{\infty} a_n$.

36.) Consider $\sum_{n=2}^{\infty} \frac{1}{n^k \ln n}$; if $\underline{\underline{k=0}}$, then $\sum_{n=2}^{\infty} \frac{1}{n^0 \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ <u>diverges</u> by problem 25 of this section ; if $\underline{\underline{k=1}}$, then $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

diverges by problem 5 in section 10.4;

assume $k<0$: then $k=-r<0$ $(r>0)$ and

$\sum_{n=2}^{\infty} \frac{1}{n^k \ln n} = \sum_{n=2}^{\infty} \frac{n^r}{\ln n}$; apply nth term test :

$$\lim_{n\to\infty} \frac{n^r}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{r n^{r-1}}{\frac{1}{n}} = \lim_{n\to\infty} r n^{r-1} \cdot n^1$$

$= \lim_{n\to\infty} r n^r = \infty \neq 0$ so $\sum_{n=2}^{\infty} \frac{1}{n^k \ln n}$ diverges;

assume $0<k<1$: limit compare to $\frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n^k \ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^k \ln n} = \lim_{n\to\infty} \frac{n^{1-k}}{\ln n}$$

(where $1-k>0$) $\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{(1-k)n^{1-k-1}}{1/n}$

$$= \lim_{n\to\infty} n(1-k)n^{1-k} = \lim_{n\to\infty} (1-k)n^{2-k} = \infty$$

so $\sum_{n=2}^{\infty} \frac{1}{n^k \ln n}$ diverges since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

assume $k>1$: then $\frac{1}{n^k \ln n} \leq \frac{1}{n^k}$ for $n \geq 3$

and $\sum_{n=3}^{\infty} \frac{1}{n^k}$ converges by p-series test

$(p=k>1)$ so $\sum_{n=2}^{\infty} \frac{1}{n^k \ln n}$ converges by comparison test.

39.) a.) $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$; $\frac{1}{1+2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series since $r=1/2$ and $-1<r<1$; thus

$\sum_{n=1}^{\infty} \frac{1}{1+2^n}$ converges by the comparison test.

41.) assume $a_n \geq 0$, $c_n \geq 0$, $\lim\limits_{n\to\infty} c_n = 0$, and $\sum\limits_{n=1}^{\infty} a_n c_n$ converges. Show $\sum\limits_{n=1}^{\infty} a_n c_n^2$

converges : $\lim\limits_{n\to\infty} c_n = 0$ means there is some integer N so that if $n > N$, then $c_n < 1$. If $c_n < 1$, then $c_n^2 < c_n$ and $a_n c_n^2 \leq a_n c_n$. Since $\sum\limits_{n=1}^{\infty} a_n c_n$ converges, $\sum\limits_{n=1}^{\infty} a_n c_n^2$ converges by the comparison test.

43.) $\sum\limits_{n=1}^{\infty} \dfrac{n+2}{n+1} \cdot \dfrac{1}{n^3} \leq \sum\limits_{n=1}^{\infty} 2 \cdot \dfrac{1}{n^3}$ and by equation ⊛) from class handout,

$$\sum_{n=1}^{\infty} \frac{2}{n^3} < \frac{2}{1^3} + \int_1^{\infty} \frac{2}{x^3}\,dx = 2 + \left.\frac{-1}{x^2}\right|_1^{\infty} = 2 + (0+1) = 3,$$

i.e., $\sum\limits_{n=1}^{\infty} \dfrac{n+2}{n+1} \cdot \dfrac{1}{n^3} < 3$.

44.) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n2^n} \leq \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} = \sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots$

$+ = \left(\tfrac{1}{2}\right)\left[1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \cdots\right] = \left(\tfrac{1}{2}\right) \cdot \dfrac{1}{1-\frac{1}{2}} = \dfrac{1}{2} \cdot 2 = 1$, i.e.,

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n2^n} < 1$.